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Abstract

Using wavelets, the author estimates the fractional order of integration of a common long-run money-demand relationship whose parameters are obtained from a full-information maximum-likelihood procedure. Because the order of integration is found to be significantly higher than zero, a grid-search procedure is used over the local parameter space to isolate the parameters required to lower the fractional order of integration. When Canadian data from 1968–99 are examined, a 25 per cent reduction in the interest semi-elasticity, accompanied by a corresponding increase in the income elasticity, is required to render the equilibrium relationship more stationary. However, given the large standard errors around the estimates of the fractional order of integration, the improvement in the estimate of the cointegration relationship is relatively modest. This suggests that money, output, prices, and interest rates are, at best, fractionally cointegrated.

JEL classification: E41, C13
Bank classification: Monetary aggregates; Econometric and statistical methods

Résumé

L'auteur a recours aux ondelettes pour estimer l'ordre d'intégration fractionnaire d'une fonction type de demande de monnaie à long terme dont il a obtenu les paramètres à l'aide d'une technique de calcul du maximum de vraisemblance à information complète. L'ordre d'intégration s'étant révélé nettement supérieur à zéro, l'auteur se livre à une recherche manuelle dans l'espace local des paramètres afin de déterminer les valeurs que ceux-ci doivent prendre pour que l'ordre d'intégration fractionnaire diminue. Lorsqu'on utilise les données canadiennes relatives à la période 1968–99, il faut réduire de 25 % la semi-élasticité par rapport au taux d'intérêt et accroître d'autant l'élasticité par rapport au revenu pour diminuer le degré de non-stationnarité de la relation d'équilibre. Toutefois, étant donné l'ampleur des écarts-types qui entourent l'estimation de l'ordre d'intégration fractionnaire, l'amélioration obtenue en ce qui concerne le degré de cointégration est relativement modeste. Ce résultat donne à penser que la monnaie, la production, les prix et les taux d'intérêt sont, au mieux, des séries cointégrées d'ordre fractionnaire.

Classification JEL : E41, C13
Classification de la Banque : Agrégats monétaires; Méthodes économétriques et statistiques

1. Introduction

Work on the demand for money experienced considerable growth in the early 1990s because of the advent of econometric techniques capable of isolating long-run equilibrium relationships in a system of economic variables. The maximum-likelihood (ML) estimator proposed by Johansen (1988) has proven particularly useful at estimating stationary (I(0)) relationships for a set of non-stationary (I(1)) variables. This two-step procedure pre-tests the orders of integration of the variables entering the system, choosing between I(1) and I(0) alternatives. It then tests for the existence of cointegration vectors among the I(1) variables, in essence determining whether a particular linear combination of the variables in the system is of lower order than the individual variables themselves.

Although popular and intuitively appealing, the two-step procedure can encounter errors in either of its steps because of its restrictive focus on I(1) and I(0) possibilities. Diebold and Rudebusch (1991) have shown that some unit root tests have low power against fractional integration alternatives (I(d), where $0 < d < 1$). In other words, the null of a unit root may be incorrectly accepted by the testing procedure, and non-I(1) variables may be included in the cointegration system. Furthermore, the test for cointegration is simply a test of I(1) versus I(0), this time on the cointegration residuals. The I(0) alternative may be too stringent, and therefore cointegration may not be found if it indeed exists. In addition, researchers may erroneously believe that they have uncovered cointegration if the variables entering the system are of a lower order than first believed. If the variables entering the system are all I(0) but are thought to be I(1), then any I(0) estimated equilibrium relationship would not prove the existence of cointegration, which by definition should be a linear combination of the variables that has an order of integration lower than the original variables. That is, the cointegration vector and original variables cannot be of the same order of integration.

Allowing for I(d) processes, in a broader definition of cointegration a linear combination of the variables can be located that has an order of integration lower than the original variables, or I(d-b) for some $b > 0$. Under this scenario, the variables are fractionally cointegrated, and the cointegration relationship has a markedly different behaviour relative to an I(0) process. In particular, if $0 < d\text{-}b < 0.5$, the cointegration relationship will be a long-memory process, and any deviations from equilibrium will be closed only in the long run. Significantly larger values than 0.5 imply that the process is covariance non-stationary, and that the return to equilibrium following a shock will be longer still.

As an example of the limitations of integer-valued orders of integration, Baillie and Bollerslev (1989) found that the exchange rates of seven industrial countries were cointegrated using an ML estimator. Diebold et al. (1994) later argued that the cointegrating relationship of Baillie and Bollerslev is not robust, because it vanishes when the constant is removed from the cointegration space. Baillie and Bollerslev (1994) reconsidered their results, and found that the equilibrium relationship in their earlier work had an order of integration not of zero but of 0.89, implying that the exchange rates are fractionally cointegrated, since each individual exchange rate was I(1). Any shock to this system will, therefore, not dissipate for a long time.

There is growing theoretical and empirical evidence that many economic variables follow I(*d*) processes. Parke (1999) states that I(*d*) processes can occur naturally when allowing for shocks that are stochastic in both magnitude and duration. Any observed value of a variable can be considered to be a weighted sum of past shocks, and these can be noticeably affected by only a few long-lasting shocks. Porter-Hudak (1990) arrives at the I(*d*) conclusion for money; Diebold and Rudebusch (1989) for output; Baillie et al. (1996) for inflation; and Backus and Zin (1993) and Tkacz (2000) for interest rates. With mounting evidence favouring I(*d*) processes for several key macroeconomic series, Barkoulas et al. (1999) argue that estimated models using such variables must take into account the fractionally integrated properties of the data, which cannot be captured with traditional linear vector autoregressions.

In light of the above evidence, we reconsider the long-run demand for money to determine whether allowing for non-integer orders of integration in the individual and joint processes changes our beliefs about the existence of stationary, long-run money-demand relationships. Much work in the area has followed the methodology of Johansen and Juselius (1990), who isolate stationary long-run money-demand relations for Finland and Denmark using an ML procedure to estimate the cointegration vectors. Hendry (1995) successfully uses this approach to estimate a long-run money-demand relationship for Canada. However, this approach does not allow for the possibility of I(*d*) processes, implying that the multivariate ML estimator of Johansen (1988) can suffer from the same limitations as the univariate Dickey-Fuller unit root tests when I(*d*) processes are permitted.

Section 2 estimates a typical long-run money-demand relationship for Canada, and the fractional order of integration of the resulting equilibrium relationship using a wavelet transformation of the autocorrelation function, as proposed by Jensen (1999). This wavelet estimator is more powerful than the frequency domain estimators typically used in the literature. Section 3 analyzes the sensitivity of individual elasticities. Using a grid search procedure, we locate the elasticities required to minimize the fractional order of integration, and find that the equilibrium relationship is most sensitive to changes in the interest semi-elasticity.

2. Traditional Money-Demand Formulation

2.1 Theory

The demand for money is well grounded in theory, where balances held for transactions are usually a function of income and the opportunity cost of holding money, proxied by a representative short-term interest rate. The long-run demand for nominal balances can be specified as

$$m_t = \varepsilon_y y_t + \varepsilon_R R_t + \varepsilon_p p_t + \eta_t \ , \tag{1}$$

where m_t is a nominal monetary aggregate, y_t the real level of aggregate income, R_t a short-term interest rate, p_t the price level, and η_t an identically, independently distributed disturbance. Lower-case variables are in logarithms, and upper-case variables are in levels. In this specification, ε_y is the long-run income elasticity of money demand, ε_R the interest semi-elasticity, and ε_p the price elasticity, all of which must be estimated.

Specification (1) has been used successfully as a foundation for error-correction models. In one of the more complete studies on money demand in Canada, Hendry (1995) estimates several M1 equations, and finds that a dummy variable accounting for structural shifts due to financial innovations in the early 1980s is required to induce long-run parameter stability. A second specification that we therefore consider is

$$m_t = \varepsilon_y y_t + \varepsilon_R R_t + \varepsilon_p p_t + \beta D_t + \eta_t \ , \tag{2}$$

where D_t is a shift dummy taking a value of 0 prior to 1982, and 1 thereafter.

Because we examine only the fractional orders of integration of typical money-demand relationships, we refrain from exhaustive specification searches.

2.2 Data

The data were obtained from CANSIM. We use nominal M1, nominal GDP, the 90-day commercial paper rate, and the consumer price index (CPI). Real GDP is obtained by deflating nominal GDP with the CPI. The sample ranges from 1Q68–4Q99, giving 128 observations.

Table 1 shows ordinary augmented Dickey-Fuller (ADF) unit root tests on each series. Consistent with previous studies, we are unable to reject the unit root null for all series. However, as stated earlier, unit root tests can have low power against fractionally integrated alternatives. In Table 2 we therefore estimate the fractional orders of integration of each series using two different methods. The first three columns estimate *d* using the wavelet ordinary least sequence (OLS) estimator of Jensen (1999). The final three columns estimate frequency domain using the estimator of Geweke and Porter-Hudak (1983), henceforth GPH. The wavelet estimator was shown by Jensen to have considerably more power than the GPH estimator and thus it is our preferred estimator; we retain estimates using the GPH because it is more widely used. We present estimates for several different wavelets and frequency ordinates to verify their robustness. The appendix discusses each estimator in detail.

The results from the wavelet estimator (Table 2) show that the nominal M1, real GDP, and CPI have the highest estimates, and are within two standard errors (an approximate 95 per cent confidence interval) of a unit root. The interest rate has noticeably lower estimates, and in two cases it is more than two standard errors away from a unit root. This is consistent with the results of Tkacz (2000), and indicates that shocks to this variable will not be infinitely lived, although they will likely take several years to dissipate. Parke's (1999) error-duration interpretation of fractional integration indicates that this process may be heavily influenced by a handful of long-lasting shocks that have occurred since 1968.

The estimates of the various fractional orders of integration using the GPH estimator are shown in the final three columns of Table 2. Corroborating the wavelet estimates, the interest rate has the lowest orders of integration. However, the large errors around these estimates do not allow us to conclude whether this series possesses a unit root. The standard errors are heavily influenced by the small number of frequency regression ordinates used in the estimation, which range from only 11–18.

2.3 Cointegration vectors

We estimate two systems, one that includes a dummy variable to capture a potential regime shift in the early 1980s, and one that does not. We use Johansen's multiple-equation ML estimator; it can estimate multiple cointegration relationships simultaneously, should they exist, and it outperforms most single-equation methods (Gonzalo (1994)).

Table 3 shows the results of the cointegration tests and the estimated cointegration vectors. Based on the Trace test, there is at most one significant cointegration vector in each system, while the λ-max test indicates that a second vector may exist in the system without a dummy. When the test results conflict it is advisable to side with the Trace test, because it is more powerful. Furthermore, the estimated elasticities in the first cointegration vectors have theoretically valid signs, and are of magnitudes that would be consistent with most long-run money-demand relationships. These vectors are prime candidates for a more extensive analysis of long-run money demand.

As stated earlier, ADF tests perform poorly if we allow for fractional cointegration under the alternative hypothesis. The Johansen ML estimator is a multivariate version of the ADF test, thus it too may be unreliable under fractional integration. Because Table 2 indicates that at least one of our variables may be fractionally integrated, scrutinization of the estimated, disequilibrium terms, or money gaps, is warranted.

Table 4 shows the results of unit root tests on the two disequilibrium terms that arise from our ML-estimated cointegration vectors. Using standard ADF critical values, the null of a unit root is rejected at the 95 per cent level for both gaps.[1] Visual inspection of the gaps, plotted in Figure 1, reveals evidence of mean-reversion. However, both gaps have been negative since 1990, indicating that mean-reversion is not rapid.

Although the unit root tests on the money gaps offer preliminary evidence as to whether the estimated relationships are mean-reverting, such tests are not conclusive in all cases. To examine more closely the mean-reversion speeds of the estimated relationships, we explicitly estimate the fractional order of integration using the wavelet and frequency domain estimators.

For the nominal money-demand systems (Table 5), the orders of integration range from I(0.69)–I(0.83), with or without a dummy. For two of the three wavelets used in the estimation, the I(1) scenario is sufficiently remote to be discarded. The magnitudes of these estimates, however, suggest that a shock to the system will take several years to dissipate, since the short-run stationary I(0) case is far more remote than the random walk I(1) case.

1. Since the money gaps are estimated series, the ordinary ADF critical values may not be appropriate. In Table 20.2 in Davidson and MacKinnon (1993), the corrected critical values imply that we would be unable to reject the unit root null.

The GPH estimates range from I(0.66)–I(0.79), but because of the large standard errors associated with these estimates we are unable to discard the unit root hypothesis.

3. Grid Search for Lowest *d*

To uncover the estimated cointegrating relationship parameters that are responsible for the persistence of shocks to the system, we examine how the money gap's fractional order of integration changes in response to movements in the elasticities around their initial Johansen estimates. We focus solely on the system without a dummy for several reasons. First, there is evidence that the estimated parameters provide a money gap that at least is mean-reverting in the long run. Second, the estimated parameters have signs and magnitudes that are not implausible with economic theory. And third, the dummy variable adds little to the estimated relationship, as demonstrated by the unit root tests, estimated orders of integration, and graphical evidence. Imposing few restrictions will allow us to further understand the data-generating process behind the long-run relationship of money, income, interest rates, and prices.

To understand the sensitivity of the money-demand system to changes in its individual elasticities, we first allow only one elasticity to vary in isolation from the initial Johansen vector. We then consider varying two elasticities simultaneously, and finally allow all elasticities to vary. By computing the fractional order of integration for each new money gap that ensues when at least one elasticity changes, we can uncover the long-run relationship that produces the most stationary money gap. A stationary money gap allows us to understand how money demand changes in response to income, interest rate, and price changes. Depending on the magnitude of the fractional order of integration, we hope that money demand will respond rapidly enough to return the system to equilibrium over a period that is useful to policy-makers (one to two years). Furthermore, a stationary money gap can provide useful information in short-run forecasting models.

3.1 One variable elasticity

We denote the elasticities by ε_i, for $i = \{y, R, p\}$, such that a cointegration relationship can be written as $[1, -\varepsilon_y, -\varepsilon_R, -\varepsilon_p]$. In Figure 2 we allow the income elasticity to deviate from its initial estimate, while the interest semi-elasticity and the price elasticity remain fixed at their initial Johansen estimates. We therefore consider the cointegrating systems $[1, -\varepsilon_y, 0.207, -0.934]$, where ε_y lies in the interval [0.500, 2.500]. The interval over which we allow the income elasticity to vary is largely driven by economic theory, as we would deem implausible any suspiciously small or large income elasticities.[2] Figure 2 shows that the fractional order of integration of the money gap varies little with income elasticity. In the base case vector we have I(0.697), which dips to I(0.585) when ε_y grows to 1.600. However, even with this large income elasticity, the confidence interval around our estimate of *d* widens noticeably, so that even the unit root hypothesis cannot be excluded. Beyond 1.600 the fractional order of integration rises again, settling around I(0.75).

Figure 3 shows that the interest semi-elasticity varies in the interval [0.000, 0.500], while the income and price elasticities remain constant. There is substantial movement in the fractional order of integration of the money gap when the interest semi-elasticity is low. At -0.02 it is above I(1), while at -0.124 it falls to I(0.539). Beyond that it settles around I(0.75). Figure 3 shows that to lower the overall order of integration, the interest rate semi-elasticity would likely require adjustment. Any non-stationarity observed in the Johansen money gap may be due to an imprecise interest semi-elasticity since, as Table 2 showed, the interest rate is the most likely variable to be fractionally integrated.

Figure 4 allows for a variable price elasticity over the interval [0.500,2.500]. We witness the same kind of “smile” that we had for the variable income elasticity. The lowest value of *d* is attained for a price elasticity of 1.253, which is associated with wide confidence bands. The fractional order of integration converges towards I(0.80) as the price elasticity increases.

3.2 Two variable elasticities

Figure 5 plots income elasticities on the x-axis, interest semi-elasticities on the y-axis, and (1-*d*) on the z-axis. We have (1-*d*) instead of *d* on the z-axis purely to improve the visual aspects of the diagrams. Maximizing (1-*d*) is equivalent to minimizing *d*, so we are searching for the highest points on the three-dimensional diagrams. A dramatic peak occurs for an income elasticity of 1.15 and an interest semi-elasticity of -0.145. The fractional order of integration of the money gap at these coordinates is I(0.179), which is the closest to stationarity that we have been thus far.

In Figure 6 we allow the income and price elasticities to vary. The smallest order of integration, I(0.572), occurs at an income elasticity of 0.75 and a price elasticity of 1.35.

2. For space limitations we present only the estimates that use the Daubechies-4 wavelet. Estimates that use other wavelets do not differ noticeably.

This improves little upon the orders of integration uncovered when only one or the other of these variables was allowed to vary in Figures 2 and 4, respectively. Fixing the interest semi-elasticity at -0.207, therefore, does not allow us to significantly lower the order of integration of the money gap.

In Figure 7 we allow the interest semi-elasticity and the price elasticity to vary. Mirroring the findings in Figure 5, there is a single dramatic peak for an interest semi-elasticity of -0.150 and price elasticity of 1.05. Apparently, the responsiveness of money demand to interest rate changes would have to be lowered by about 25 per cent, from about -0.20 to about -0.15, for the stationarity of the money gap to be improved.

3.3 Three variable elasticities

In this least restrictive, and most exhaustive, of searches, we allow all three elasticities to deviate freely from their initial Johansen ML estimates. Being unable to plot the results, we present the vector that produces the lowest fractional order of integration. Some 180,000 possible gaps were explored, and the cointegration relationship with the lowest order of integration was as follows:

$$[1, -\varepsilon_y, -\varepsilon_R, -\varepsilon_p] = [1, -1.205, 0.145, -0.908].$$

The estimated order of integration of the disequilibrium term arising from the above equilibrium relationship is only I(0.022). However, a substantial standard error of 0.802 makes a precise inference concerning stationarity difficult. Furthermore, the money gap from this relationship is entirely negative, implying that a constant is required in the cointegration space to have this relationship fluctuate around zero. After correcting for the long-run mean of this gap (-2.644), we obtain the money gap plotted in Figure 8. Compared to the original Johansen money gap, the gap that minimizes d is less volatile, implying that deviations from equilibrium are less pronounced.

It is likely that these variables are fractionally cointegrated, since no linear relationships appear to exist between these variables that are pure I(0) processes. Estimates of money gaps that are near I(0) are associated with very high standard errors, making inference difficult. However, a number of relationships have been estimated that are around I(0.50), and as long as any of them are significantly lower than the fractional order of integration of the interest rate, which is around I(0.80), fractional cointegration would exist.

4. Conclusion

This paper extends the long-run analysis of money demand into the sphere of fractional cointegration. This extension is motivated by the fact that traditional methods of cointegration analysis yield relationships that are not as stationary as they are believed to be. Our results demonstrate that money gaps arising from simple money-demand systems are at best fractionally integrated, and at worst random walks.

To find stationary long-run relationships, we use grid searches to isolate the parameters that minimize the fractional order of integration of the money gap. Relative to the initial Johansen ML estimates, the interest semi-elasticity must be about 25 per cent lower, while the income elasticity must be about 10 per cent higher. The price elasticity is relatively unchanged, remaining just below one. The money gap arising from this analysis, however, is associated with a wide confidence band. The order of integration seems to be most sensitive to changes in the interest semi-elasticity in the range from zero to -0.20. Future work should examine why the demand for M1 is sensitive to changes in the interest semi-elasticity when the latter is low. A threshold model, allowing for different interest semi-elasticities in low and high interest rate regimes, may allow some of this instability to be captured.

From a policy-making perspective, any advice that stems from the use of money gaps should focus on the relevant horizon. The speed with which such gaps close is slow, and if the fractional order of integration is high and not statistically different from one, we cannot accurately predict the time it will take for the gap to vanish. This problem can be exacerbated if further financial innovations take place in the economy.

Table 1

ADF unit root tests on data

Variable	Lags	Statistic
Sample: 1968Q1–1999Q4		
Nominal M1 m_t	6	-2.9938 (-3.4472)
Real GDP y_t	9	-2.3971 (-3.4481)
Interest rate R_t	1	-2.5043 (-3.4458)
CPI p_t	12	-0.1957 (-3.4491)

Constant and trend are included in the ADF regressions. Lags to correct for serial correlation are chosen by minimizing Akaike's information criterion, allowing for a maximum of 12 lags. Critical values in parentheses are from MacKinnon (1991), and are used to test the unit root hypothesis at the 95 per cent significance level. All variables, except for the interest rate, are in logarithms.

Table 2

Estimated fractional orders of integration of data

Sample: 1968Q1–1999Q4						
Variable	Wavelet domain estimates Jensen (1999)			Frequency domain estimates Geweke and Porter-Hudak (1983)		
	Daubechies-4	Daubechies-12	Daubechies-20	$T^{0.5}$	$T^{0.55}$	$T^{0.60}$
Nominal M1 m_t	0.8910 (0.0515)	0.9169 (0.0600)	0.9045 (0.0435)	0.9891 (0.2725)	0.9825 (0.2302)	0.9779 (0.1945)
Real GDP y_t	0.8701 (0.0718)	0.9078 (0.0717)	0.9047 (0.0483)	0.9724 (0.2725)	0.9752 (0.2302)	0.9598 (0.1945)
Interest rate R_t	0.7746* (0.0978)	0.8721 (0.0649)	0.8287* (0.0769)	0.5758 (0.2725)	0.5591 (0.2302)	0.7339 (0.1945)
CPI p_t	0.9539 (0.0760)	0.9922 (0.0718)	0.9828 (0.0546)	1.0775 (0.2725)	1.0500 (0.2302)	1.0405 (0.1945)

Estimates from the wavelet domain are computed using the Daubechies wavelet and three different levels of smoothing. The frequency domain estimates are computed for three different numbers of frequency, which are powers of the number of observations in the sample, T=128. Standard errors are in parentheses. (*) indicates that the variable is more than two standard errors from a unit root.

Table 3

Johansen cointegration vectors

Sample: 1968Q1–1999Q4			
System	Cointegration vectors	λ-max	Trace
$[m_t, y_t, R_t, p_t]$	[1, -0.958, 0.207, -0.934]	42.34* 19.24*	66.24* 23.90
$[m_t, y_t, R_t, p_t, D_t]$	[1, -1.041, 0.207, -0.700, -0.246]	51.73* 18.84	85.82* 34.09

Four lags, corresponding to the frequency of the data, are used for each system. (*) indicates that the statistic for existence of the cointegration vector is significant at the 95 per cent level.

Table 4

ADF unit root tests on money gaps

Sample: 1968Q1–1999Q4		
System	AIC (Lags)	BIC (Lags)
$[m_t, y_t, R_t, p_t]$	-2.492* (1)	-2.492* (1)
$[m_t, y_t, R_t, p_t, D_t]$	-2.541* (1)	-2.541* (1)

No constant and no trend are included in the ADF regressions. (*) indicates that the unit root null can be rejected at the 95 per cent level.

Table 5

Estimated fractional orders of integration, money gaps

Sample: 1968Q1–1999Q4						
System	Wavelet domain estimates Jensen (1999)			Frequency domain estimates GPH (1983)		
	Daubechies-4	Daubechies-12	Daubechies-20	$T^{0.5}$	$T^{0.55}$	$T^{0.60}$
$[m_t, y_t, R_t, p_t]$	0.697*	0.817	0.826*	0.660	0.681	0.732
$[m_t, y_t, R_t, p_t, D_t]$	0.692*	0.823	0.831*	0.795	0.788	0.793

Estimates from the wavelet domain are computed using the Daubechies wavelet with three different levels of smoothing. The frequency domain estimates are computed for three different numbers of frequency, which are powers of the number of observations in the sample, T=128. (*) indicates that d is more than two standard errors below 1, roughly equivalent to a 95 per cent confidence level.

Figure 1: Johansen money gaps

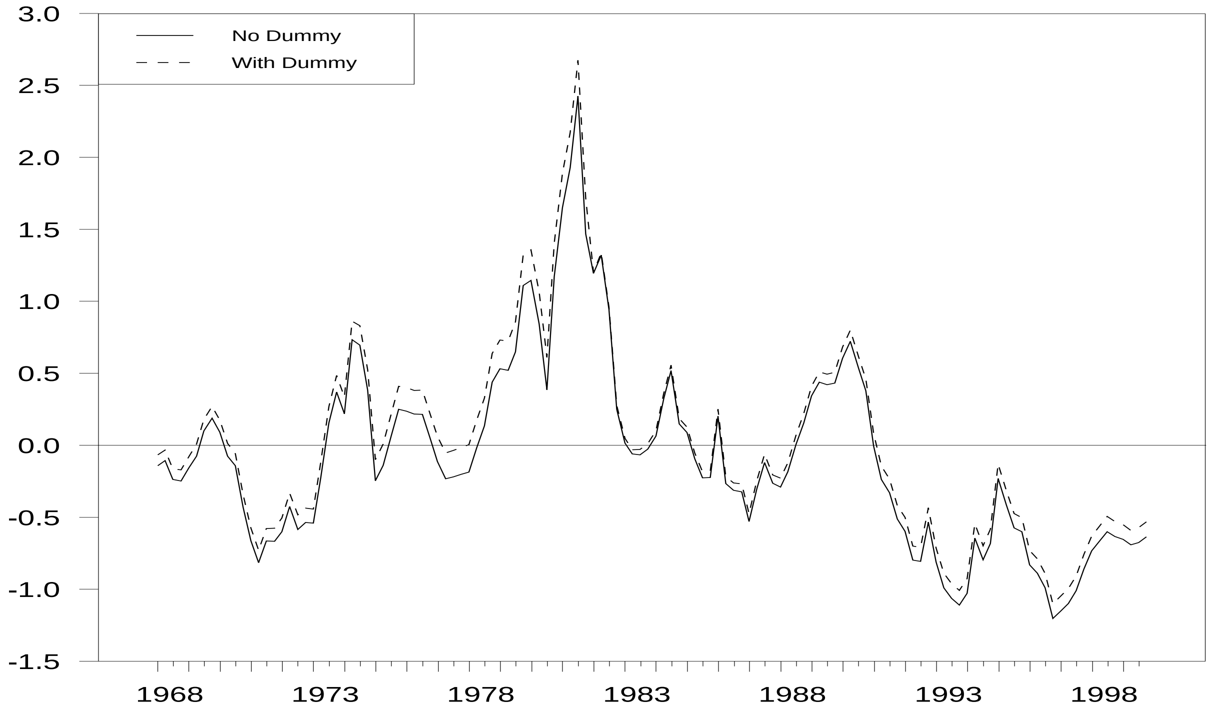
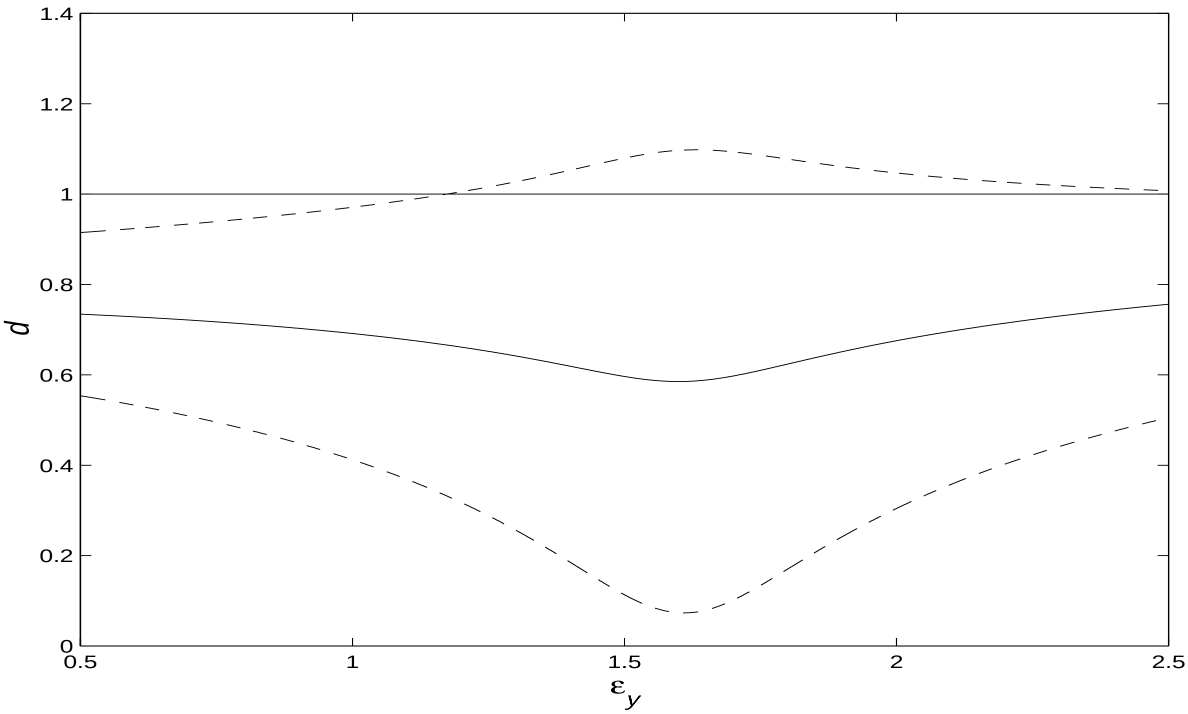


Figure 2: ε_y **and** d **for Johansen money gap**

System [m_t, y_t, R_t, p_t]: [1, $-\varepsilon_y$, 0.207, -0.934]



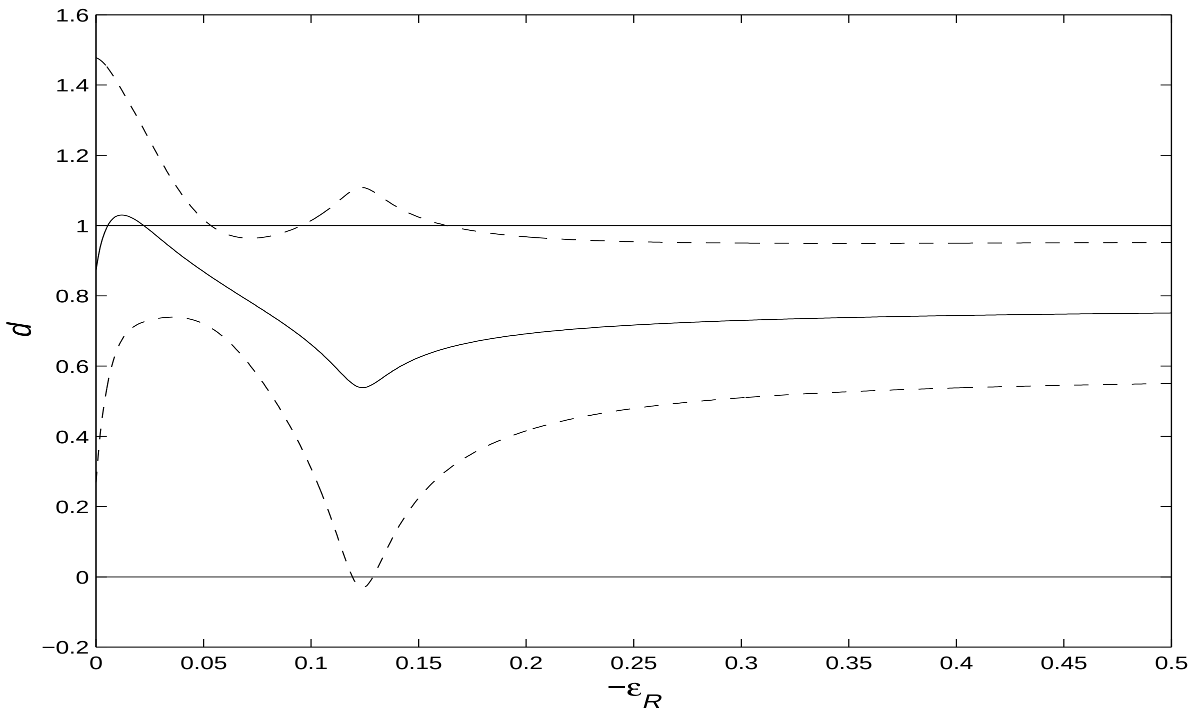
Search for income elasticity yielding lowest fractional order of integration for the money gap is found using a grid search in the interval [0.500,2.500] with steps equalling 0.001. The interest and price elasticities are kept constant at their Johansen ML-estimated values. The Daubechies-4 wavelet is used to estimate *d*. Dashed lines are a 95 per cent confidence interval.

Original Johansen vector: [1, -0.958, 0.207, -0.934] with d = 0.697.

Vector with variable income elasticity and lowest *d*: [1, -1.600, 0.207, -0.934] with d = 0.585.

Figure 3: ε_R and d for Johansen money gap

System $[m_t, y_t, R_t, p_t]$: $[1, -0.958, -\varepsilon_R, -0.934]$



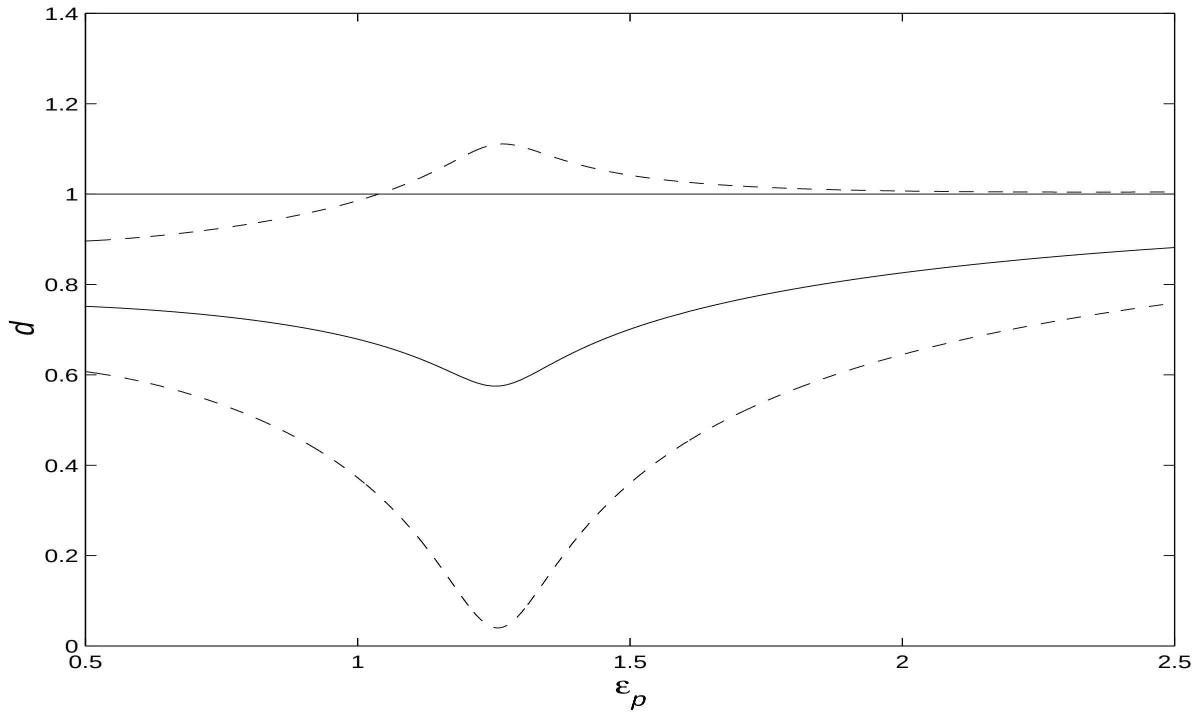
Search for interest semi-elasticity yielding lowest fractional order of integration for the money gap is found using a grid search in the interval [0.000,0.500] with steps equalling 0.001. The income and price elasticities are kept constant at their Johansen ML-estimated values. The Daubechies-4 wavelet is used to estimate *d*. Dashed lines are a 95 per cent confidence interval.

Original Johansen vector: [1, -0.958, 0.207, -0.934] with $d = 0.697$.

Vector with variable interest semi-elasticity and lowest *d*: [1, -0.958, 0.124, -0.934] with $d = 0.539$.

Figure 4: ε_p and d for Johansen money gap

System [m_t, y_t, R_t, p_t]: [1, –0.958, 0.207, $-\varepsilon_p$]



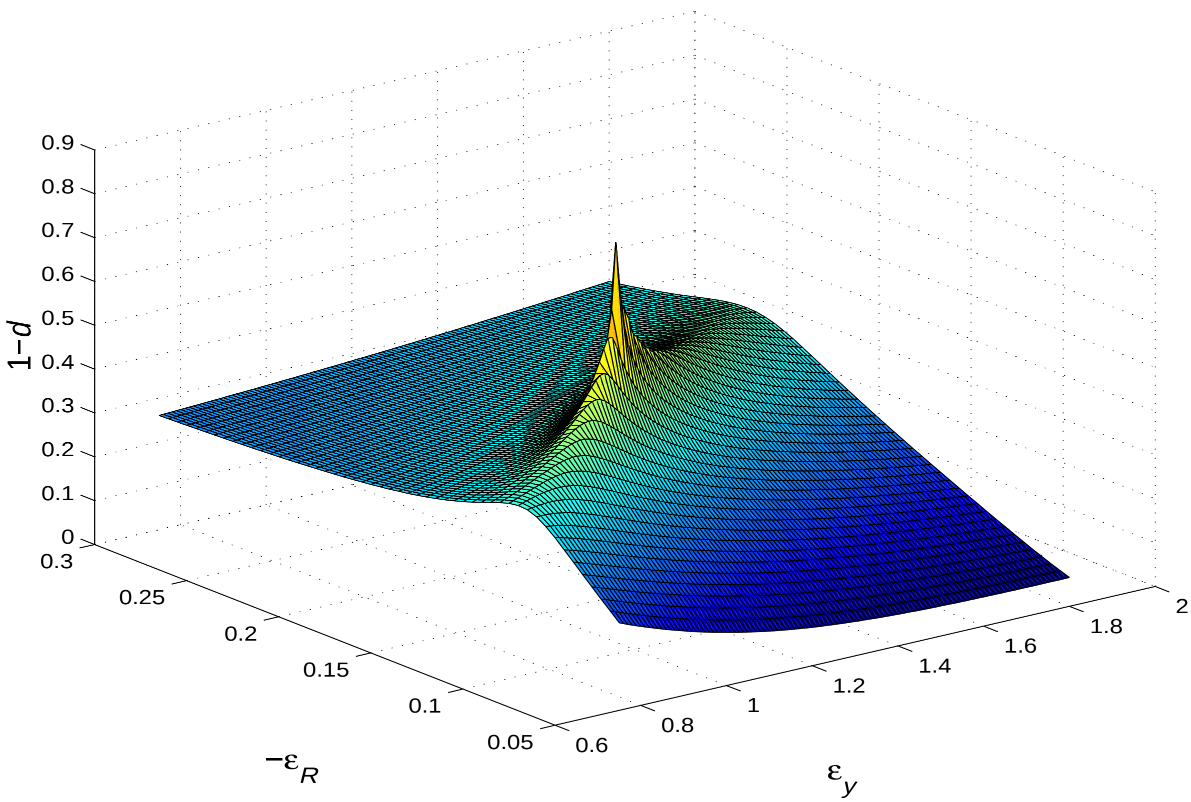
Search for price elasticity yielding lowest fractional order of integration for the money gap is found using a grid search in the interval [0.500,2.500] with steps equalling 0.001. The income and interest elasticities are kept constant at their Johansen ML-estimated values. The Daubechies-4 wavelet is used to estimate *d*. Dashed lines are a 95 per cent confidence interval.

Original Johansen vector: [1, -0.958, 0.207, -0.934] with d = 0.697.

Vector with variable price elasticity and lowest *d*: [1, -0.958, 0.207, -1.253] with d = 0.575.

Figure 5: (ε_y, ε_R) and d for Johansen money gap

System [m_t, y_t, R_t, p_t]: [1, $-\varepsilon_y$, $-\varepsilon_R$, -0.934]



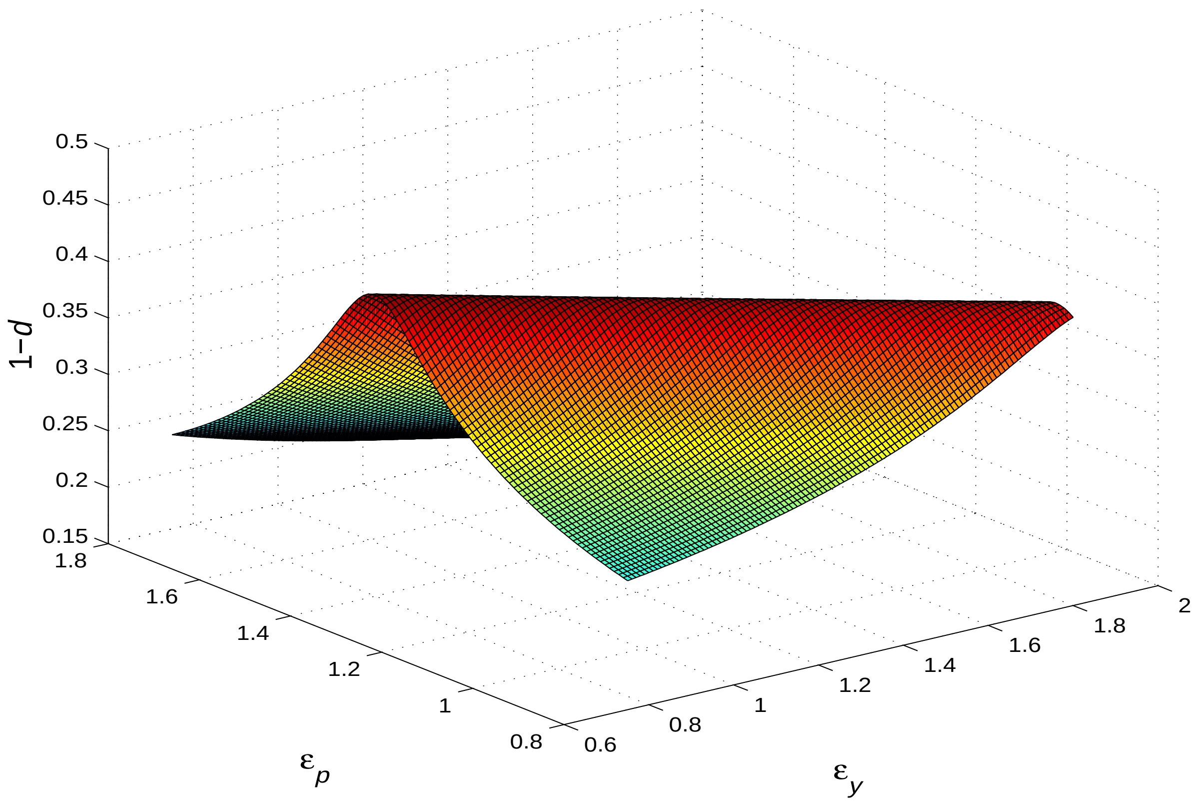
Search for income elasticity and interest semi-elasticity yielding lowest fractional order of integration (d) for the money gap is found using a grid search in the interval [0.75,1.80] for the income elasticity and [0.05,0.30] for the interest semi-elasticity, with steps equalling 0.01 for the income elasticity and 0.005 for the interest semi-elasticity. The price elasticity is kept constant at its Johansen ML-estimated value. The Daubechies-4 wavelet is used to estimate d.

Original Johansen vector: [1, -0.958, 0.207, -0.934] with d = 0.697.

Vector with fixed price elasticity and lowest d: [1, -1.150, 0.145, -0.934] with d = 0.179.

Figure 6: (ε_y, ε_p) and d for Johansen money gap

System $[m_t, y_t, R_t, p_t]$: $[1, -\varepsilon_y, 0.207, -\varepsilon_p]$



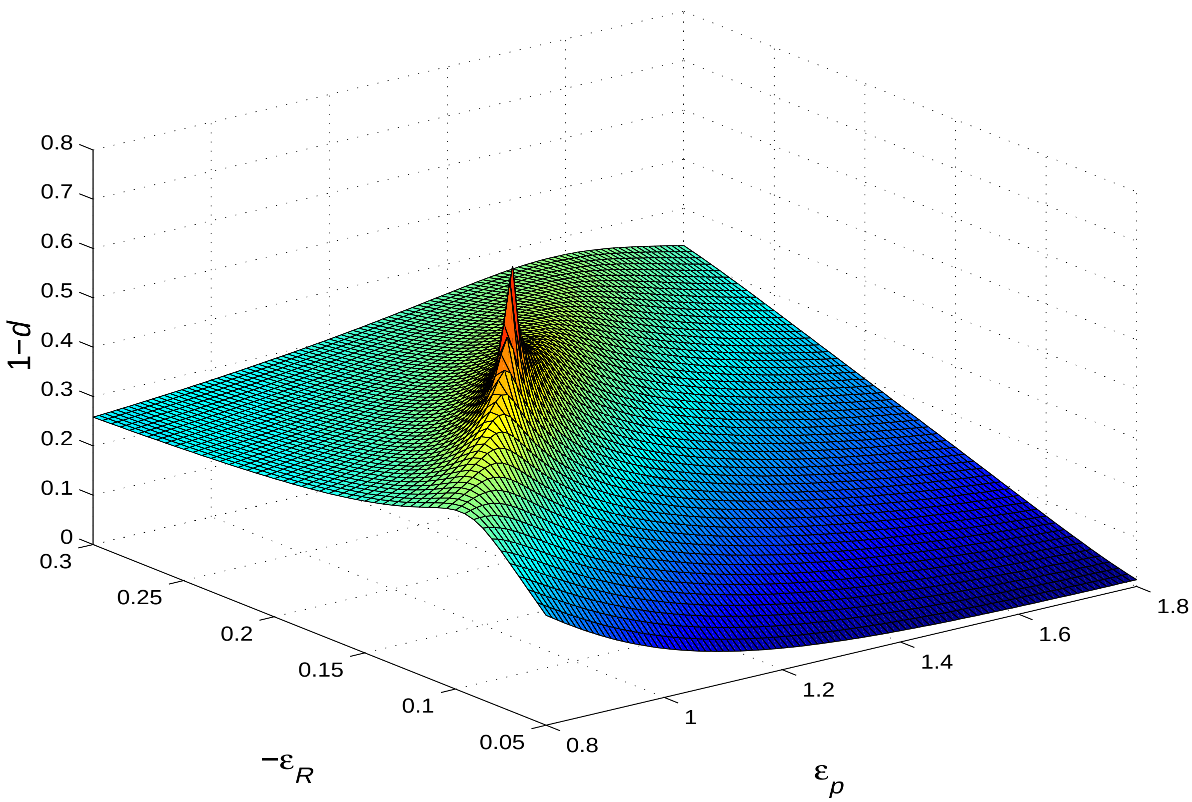
Search for income and price elasticities yielding lowest fractional order of integration (d) for the money gap is found using a grid search in the interval [0.75,1.80] for the income elasticity and [0.80,1.80] for the price elasticity, with steps equalling 0.01. The interest semi-elasticity is kept constant at its Johansen ML-estimated value. The Daubechies-4 wavelet is used to estimate d.

Original Johansen vector: [1, -0.958, 0.207, -0.934] with d = 0.697.

Vector with fixed interest semi-elasticity and lowest d: [1, -0.750, 0.207, 1.35] with d = 0.572.

Figure 7: (ε_R, ε_p) and d for Johansen money gap

System [m_t, y_t, R_t, p_t]: [1, -0.958, $-\varepsilon_R$, $-\varepsilon_p$]

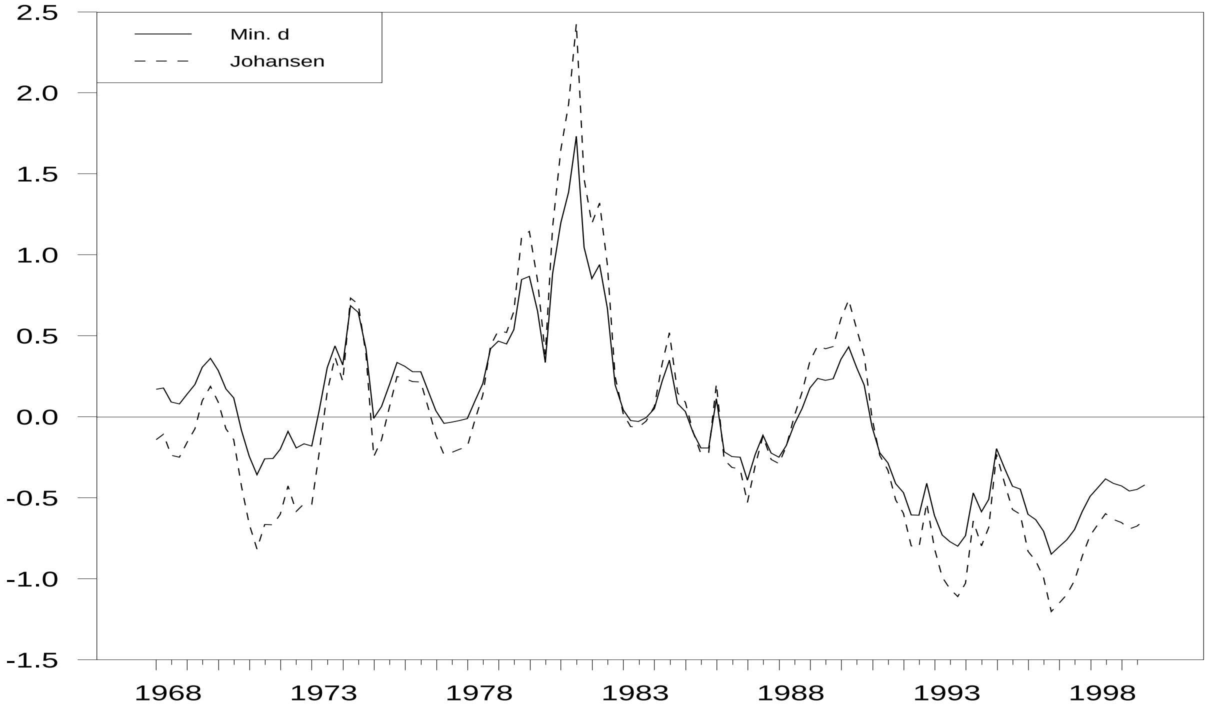


Search for interest semi-elasticity and price elasticity yielding lowest fractional order of integration (d) for the money gap is found using a grid search in the interval [0.050, 0.300]] for the interest semi-elasticity and [0.80,1.80] for the price elasticity, with steps equalling 0.005 for the interest semi-elasticity and 0.01 for the price elasticity. The income elasticity is kept constant at its Johansen ML-estimated value. The Daubechies-4 wavelet is used to estimate d.

Original Johansen vector: [1, -0.958, 0.207, -0.934] with d = 0.697.

Vector with fixed income elasticity and lowest d: [1, -0.958, 0.150, -1.050] with d = 0.286.

Figure 8: Money gap with lowest *d*



Search for elasticities yielding lowest fractional order of integration (*d*) for the money gap is found using a grid search in the interval [0.900,1.300] for the income elasticity, [0.010, 0.200] for the interest semi-elasticity, and [0.900,1.300] for the price elasticity, with steps equalling 0.001. A constant of 2.644 is used to scale the *Min. d* money gap upwards. The Daubechies-4 wavelet is used to estimate *d*.

Original Johansen vector: [1, -0.958, 0.207, -0.934] with $d = 0.697$.

Vector with lowest *d*: [1, -1.205, 0.145, -0.908] with $d = 0.022$.

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Appendix
Estimators of *d*

Consider that a unit root process x_t has the following representation:

$$(1-L)x_t = \psi(L)\varepsilon_t , \tag{A1}$$

where L is the lag operator such that $Lx_t = x_{t-1}$, and $\psi(L)\varepsilon_t$ is the moving average part. Suppose that we generalize the process to allow for fractional integration:

$$(1-L)^d x_t = \psi(L)\varepsilon_t \quad , \tag{A2}$$

where d may not necessarily equal 1. Geweke and Porter-Hudak (1983) propose a method to estimate d in the frequency domain. We begin by rewriting the left-hand side of (A2) as

$$(1-L)^d x_t = (1-L)(1-L)^{d-1}x_t = (1-L)^{\tilde{d}}z_t , \tag{A3}$$

where $z_t = (1-L)x_t$ and $\tilde{d} = d-1$. Combining (A3) and the right-hand side of (A2) we then have

$$(1-L)^{\tilde{d}}z_t = \psi(L)\varepsilon_t = u_t , \tag{A4}$$

with u_t denoting a stationary process. Equation (A4) represents an ARFIMA (fractionally integrated ARMA) model. We can estimate the fractional integration parameter by deriving the population spectrum of z_t, $f_z(\vartheta)$, which is

$$f_z(\vartheta) = \left|1-\exp(-i\vartheta)\right|^{-2\tilde{d}} f_u(\vartheta) = \left[4\sin\sin\left(\frac{\vartheta}{2}\right)\right]^{-\tilde{d}} f_u(\vartheta) \ , \tag{A5}$$

where $f_u(\vartheta)$ is the population spectrum of the stationary process u_t, and i is an imaginary number such that $i = \sqrt{-1}$. Taking logarithms we then have

$$\ln\{f_z(\vartheta)\} = \ln\{f_u(\vartheta)\} - \tilde{d}\ln\left\{4\sin\sin\left(\frac{\vartheta}{2}\right)\right\} . \tag{A6}$$

Hence, we regress the periodogram $I_T(\vartheta_j)$ at frequencies $\vartheta_j = (2\pi j)/T$, where $0 < k_1 \le j \le K < T$, such that T is the total number of observations, K the total number of spectral ordinates ($T^{0.5}$, $T^{0.55}$, or $T^{0.60}$), and k_1 the first spectral ordinate, against a constant and $\ln\{4\sin\sin(\vartheta_j/2)\}$. This provides an estimate of $\tilde{d}$, and hence d.

A recently developed alternative to Fourier transforms are wavelet transforms, where a given function $f(x)$ can be expressed in the wavelet domain in the following manner:

$$f(x) = c_0 + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} c_{jk} \psi(2^j x - k), \tag{A7}$$

with $\psi(x)$ defined as

$$\psi(x) = \begin{cases} 1 \text{ if } 0 \le x < \frac{1}{2} \\ -1 \text{ if } \frac{1}{2} \le x < 1 \\ 0 \text{ otherwise} \end{cases} . \tag{A8}$$

The group of functions $\psi_{jk}(x) = \psi(2^j x - k)$ for $j \ge 0$ and $0 \le k < 2^j$ are orthogonal, and collectively form a basis in the space of all square-integrable functions $\mathbf{L}_2$ along the [0,1] interval. The index j is the dilation (or scaling) index, which compresses the function $\psi(x)$, and the index k is the transition index that shifts the function $\psi(x)$. Generally, any such basis in $\mathbf{L}_2(\mathbf{R})$ is called a wavelet, and (A8) is called a Haar wavelet.

Jensen (1999) demonstrates that, for an I(d) process x_t with $|d| < \frac{1}{2}$, use of the autocovariance function implies that the wavelet coefficients c_{jk} in (A7) are distributed as $N(0, \sigma^2 2^{-2jd})$. If $R(j)$ denotes the wavelet coefficient's variance at scale j, then after taking logarithms an estimate of d can be obtained from

$$\ln R(j) = \ln \sigma^2 - d \ln 2^{2j} . \tag{A9}$$

The wavelets are used only in the estimation of the d consistent with the observed autocovariance function. Furthermore, because of the form of the wavelet expansion (A7), the number of observations for the underlying process x_t must be a factor of 2. Several different wavelets have been proposed, and they usually involve smoothing the step function (A8). The Daubechies (1988) wavelet is an example of such a smooth wavelet, and is used in our applications. The degree of smoothing increases with the order of the wavelet, such that the Daubechies-20 wavelet is smoother than the Daubechies-12 wavelet, which in turn is smoother than the Daubechies-4 wavelet. There is no universal agreement on the optimal amount of smoothing for wavelets, a situation akin to the amount of smoothing required for non-parametric kernel regressions. We used the Daubechies wavelet for two reasons: it is used in many applications outside economics, and Jensen (1999) demonstrated the desirable properties of the wavelet estimator using this wavelet. Several alternative wavelets are described in Vidakovic (1999).

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