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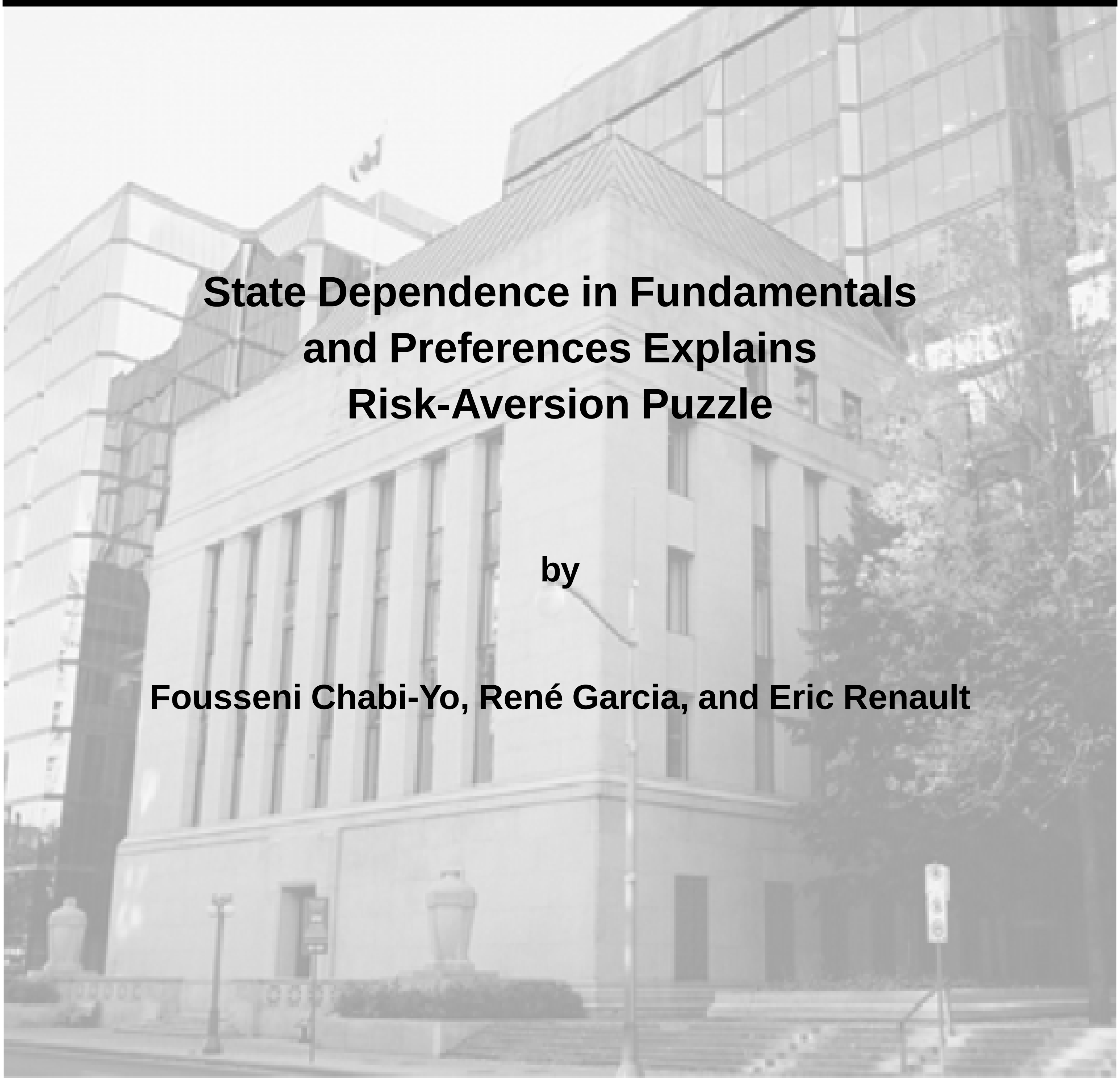

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Abstract

The authors examine the ability of economic models with regime shifts to rationalize and explain the risk-aversion and pricing-kernel puzzles put forward in Jackwerth (2000). They build an economy where investors' preferences or economic fundamentals are state-dependent, and simulate prices for a market index and European options on that index. Based on the original non-parametric methodology, the risk-aversion and pricing-kernel functions obtained across wealth states with these artificial data exhibit the same puzzles found with the actual data, but within each regime the puzzles disappear. This suggests that state dependence potentially explains the puzzles.

JEL classification: G12, G13
Bank classification: Financial markets; Market structure and pricing

Résumé

Les auteurs présentent un modèle économique à changement de régime qui permet de reproduire les énigmes relatives à l'aversion pour le risque et au facteur d'actualisation stochastique mises en évidence par Jackwerth (2000). Ils construisent un modèle où les préférences des investisseurs et leur consommation dépendent d'une variable d'état qui suit un processus de type markovien à deux régimes et génèrent une série de prix d'options d'achat européennes. Au moyen de la méthodologie d'estimation non paramétrique proposée par Jackwerth, ils déduisent des fonctions d'aversion absolue pour le risque et d'actualisation stochastique pour chaque valeur de la richesse. Ces fonctions présentent les mêmes anomalies que celles obtenues par Jackwerth à partir des données réelles. Lorsque la même méthodologie est appliquée à chaque état de l'économie, les anomalies disparaissent. D'après ces résultats, l'existence de changements de régime dans l'économie pourrait expliquer ces deux énigmes.

Classification JEL : G12, G13
Classification de la Banque : Marchés financiers; Structure de marché et fixation des prix

1. Introduction

Recently, Jackwerth (2000) and Aït-Sahalia and Lo (2000) have proposed non-parametric approaches to recover risk-aversion functions across wealth states from observed stock and options prices. In a complete market economy, which implies the existence of a representative investor, absolute risk aversion (ARA) can be evaluated for any state of wealth in terms of the historical and risk-neutral distributions. To obtain the historical distribution, Jackwerth (2000) applied a non-parametric kernel density approach to a time series of returns on the Standard & Poor's (S&P) 500 index. The risk-neutral distribution is recovered from prices on European call options written on the S&P 500 index by applying a variation of the non-parametric method introduced in Jackwerth and Rubinstein (1996). The basic idea of this method is to search for the smoothest risk-neutral distribution, which at the same time explains the options prices.

Using options prices and realized returns, Jackwerth (2000) and Jackwerth and Rubinstein (2004) find estimated values for the ARA that are nearly consistent with economic theory before the 1987 crash. However, for the post-crash period, Jackwerth (2000) finds that the implied ARA function is negative around the mean and increasing for larger wealth levels. This empirical feature, called the risk-aversion puzzle by Jackwerth (2000), has also been documented by Aït-Sahalia and Lo (2000). Another way to express this puzzling result is through the pricing kernel across wealth states. A pricing-kernel puzzle occurs when the ratio of the state-price density to the historical density increases with wealth (see Brown and Jackwerth 2000). After examining several potential explanations, Jackwerth (2000) concludes that these puzzling results are most probably due to the mispricing of some options by the market.

In this paper, we propose another explanation based on the existence of state dependence in preferences or in economic fundamentals. Garcia, Luger, and Renault (2001) propose a general pricing model where the pricing kernel depends on some latent state variables, observed only by the investor. This phenomenon can be understood in either of two possible ways: (i) as in Melino and Yang (2003), investors' preferences are state-dependent; or, (ii) as in Garcia, Luger, and Renault (2003), the joint process of consumption and dividends follows a Markov-switching regime distribution such that the current regime is known only to the investors. In this paper, we use the models developed in Garcia, Luger, and Renault (2003) and Melino and Yang (2003) to generate artificial prices for stocks and options. To recover the risk-neutral distribution, we develop a simple simulation method to create a bid-ask spread around options prices and apply the same non-parametric methodology as Jackwerth and Rubinstein (1996). The historical distribution is estimated based on a mixture of lognormals. In our model, by construction, the risk-aversion functions are consistent with economic theory within each regime, since the regime is observed by investors. As in Jackwerth (2000), however, we obtain negative estimates of the risk-aversion function in some states of wealth. The pricing-kernel function across wealth states also exhibits a puzzle, even though this function is decreasing within each regime. We therefore provide another potential explanation for the puzzles put forward by Jackwerth (2000).

The remainder of this paper is organized as follows. In section 2, we present Jackwerth's (2000) approach for recovering the ARA function across wealth states. In section 3, we build a utility-based economic model with state dependence in preferences and endowments, and describe how to simulate artificial options and stock prices in this economy. In section 4, we recover the risk-aversion and pricing-kernel functions across wealth states and discuss the results. In section 5, we offer some conclusions.

2. The Pricing-Kernel and Risk-Aversion Puzzles

In this section, we outline the puzzles put forward by Jackwerth (2000) as well as the methodology used to exhibit those puzzles.

2.1 Theoretical underpinnings

Under very general non-arbitrage conditions (Hansen and Richard 1987), the time t price of an asset that delivers a payoff g_{t+1} at time $(t+1)$ is given by:

$$p_t = E_t\left[m_{t+1}g_{t+1}\right], \tag{1}$$

where $E_t[.]$ denotes the conditional expectation operator given investors' information at time t. Any random variable m_{t+1} consistent with (1) is called an admissible stochastic discount factor (SDF), or pricing kernel. Among the admissible SDFs, only one, denoted by m^*_{t+1}, is a function of available payoffs. It is the orthogonal projection of any admissible SDF on the set of payoffs. Suppose some rational investor is able to separate its utility over current and future values of consumption:

$$U\left[C_t, C_{t+1}\right] = u(C_t) + \beta u\left(C_{t+1}\right). \tag{2}$$

The first-order condition for an optimal consumption and portfolio choice will imply that m^*_{t+1} coincides with the projection of $\beta\frac{u\prime(C_{t+1})}{u\prime(C_t)}$ on the set of payoffs. Therefore, through a convenient aggregation argument, the concavity of utility functions should imply that m^*_{t+1} is decreasing in current wealth.

Moreover, as Hansen and Richard (1987) show, no arbitrage implies almost sure positivity of m^*_{t+1}. Therefore, $m^*_{t+1}/E_t m^*_{t+1}$ can be interpreted as the density function of the risk-neutral probability distribution with respect to the historical one. In the case of a representative investor with preferences consistent with (2), we deduce:

$$\frac{m_{t+1}^*}{E_t m_{t+1}^*} = \frac{u'(C_{t+1})}{E_t u'(C_{t+1})}.$$

Therefore,

$$\frac{\partial Log m_{t+1}^*}{\partial C_{t+1}} = \frac{u''(C_{t+1})}{u'(C_{t+1})} \tag{3}$$

is the inverse of the Arrow-Pratt index of the ARA of the investor.

2.2 The puzzles

For the sake of simplicity, it is convenient to analyze these puzzles in a finite state-space framework. If $j = 1, \cdots, n$ denotes the possible states of nature, we get the density function of the risk-neutral distribution probability with respect to the historical one as:

$$\frac{m_{t+1}^*}{E_t m_{t+1}^*} = \frac{p_j^*}{p_j} \text{ in state } j, \tag{4}$$

where p_j^* is the risk-neutral probability across wealth states $j = 1, \cdots, n$ and p_j is the corresponding historical probability. Brown and Jackwerth (2000) use (4) to empirically derive the pricing-kernel function from realized returns on the S&P 500 index and options prices on the index over a post-1987 period. For the centre wealth states (over the range of 0.97 to 1.03, with average wealth normalized to one), they find a pricing-kernel function that is increasing in wealth. This is the so-called pricing-kernel puzzle.

As explained in section 2.1, the increasing nature of function (1) in wealth is puzzling because it is akin to a convex utility function for a representative investor, which is obviously inconsistent with the general assumption of risk aversion. From (3), the ARA coefficient can actually be computed through a log-derivative of the pricing kernel. By using (4), we deduce:

$$ARA = -\frac{u''(C_{t+1})}{u'(C_{t+1})} = \frac{p_j'}{p_j} - \frac{p_j^{*\prime}}{p_j^*}, \tag{5}$$

where p_j' and $p_j^{*\prime}$ are the derivatives of p_j and p_j^* with respect to aggregate wealth in state j.

Jackwerth (2000) observes that the ARA functions computed from (5) dramatically change shape around the 1987 crash. Prior to the crash, they are positive and decreasing in wealth, which is consistent with standard assumptions made in economic theory about investors' preferences. After the crash, they are partially negative and increasing (see Figure 3 in Jackwerth 2000). This result is called the risk-aversion puzzle. One component of it is equivalent to the pricing-kernel puzzle: ARA should be positive, because the pricing kernel should be decreasing in aggregate wealth. Additionally, even when there is no pricing-kernel puzzle (positive ARA), there remains a risk-aversion puzzle when ARA is increasing in wealth. While the pricing-kernel puzzle is observed for only (the centre) wealth states, the risk-aversion puzzle (increasing ARA) remains for larger levels of wealth. Without any discretization of wealth states, Aït-Sahalia and Lo (2000) document similar empirical puzzles for implied risk aversion.

2.3 Statistical methodology

It is possible to use several statistical methodologies to recover the historical distribution of future returns (on the underlying index), given current returns. As Jackwerth (2000) emphasizes, the choice of a particular estimation strategy should not have any impact on the documented puzzles. For instance, a kernel estimation will be valid under very general stationarity and mixing conditions.

While historical probabilities, p_j, are recovered from a time series of underlying index returns, risk-neutral probabilities, p_j^*, will be backed out of cross-sections from a set of observed options prices written on the same index. Concerning the latter, in a pioneering article, Jackwerth and Rubinstein (1996) recommend solving the following quadratic program:

$$\begin{gathered} \min_{p^*} \sum_{j=1}^{n} \left(p_j^* - \bar{p}_j\right)^2 \\ \sum_{j=1}^{n} p_j^* = 1, \quad p_j^* \geq 0, \\ C_i^* = \frac{1}{R_f} \sum_{j=1}^{n} p_j^* \max\left[0, S_j - K_i\right], \\ \frac{1}{R_f} \sum_{j=1}^{n} p_j^* S_j = S_0, \\ C_{ib}^* \leq C_i^* \leq C_{ia}^* \text{ for } i = 1,...,m \text{ and } S_b \leq S_0 \leq S_a, \end{gathered} \tag{6}$$

where C_{ib}^* (C_{ia}^*) represents the call options bid (ask) price with strike price K_i. The bid and ask stock prices are, respectively, S_b and S_a. In other words, the implied risk-neutral probabilities, p_j^*, are the closest to the prior ones, $\bar{p}_j$, that result in options and underlying asset values that fall between the respective bid and ask prices. As Jackwerth and Rubinstein (1996) emphasize, this methodology has the virtue that general arbitrage opportunities do not exist if and only if there is a solution. This remark is still valid when considering alternative quadratic programs based on other distances. For instance, Jackwerth and Rubinstein (1996) put forward the goodness-of-fit approach:

$$\min_{p^*} \sum_{j=1}^{n} \frac{\left(p_j^* - \bar{p}_j\right)^2}{\bar{p}_j}, \tag{7}$$

whereas, following Hansen and Jagannathan (1997), one may prefer:

$$\min_{p^*} \sum_{j=1}^{n} \frac{\left(p_j^* - \bar{p}_j\right)^2}{p_j}, \tag{8}$$

since, with obvious notations, the objective function (8) can be seen as $E_t(m_{t+1}^* - \overline{m}_{t+1})^2$.

Jackwerth and Rubinstein (1996), however, observe that the implied distributions are rather independent of the choice of the objective function when a sufficiently high number of options are available.[1]

Since we wish to focus on a simulation exercise, we will choose 50 options in cross-section to be sure that the solution is determined by the constraints (options and underlying asset values between bid and ask prices), and not by the objective function. In particular, the choice of the prior is immaterial and, as Jackwerth and Rubinstein (1996) note, even a pure smoothness criterion independent of any prior would do the job. They consider, in particular:

$$\min_{p^*} \sum_{j=1}^{n} \left(p_{j-1}^* + p_{j+1}^* - 2p_j^*\right)^2, \tag{9}$$

when the states $j = 1, 2 \cdots, n$ are ranked in order of increasing wealth. To remain true to the traditional approach, however, in section 4 we will use the goodness-of-fit criterion in the simulation. Prior risk-neutral probabilities, $\bar{p}_j$, will be computed, according to Breeden and Litzenberger's (1978) methodology, from second-order derivatives of options prices with respect to the strike price. Note that a necessary source of difference between p_j^* and $\bar{p}_j$ is the discretization of the state-space performed to define p_j^*.

3. Economies with Regime Shifts

In this section, we construct economies with regime shifts in endowments or preferences to simulate artificial stock and options prices.

[1] They notice that "as few as 8 option prices seem to contain enough information to determine the general shape of the implied distribution" and that "at the extreme, the constraints themselves will completely determine the solution."

3.1 The general framework

Consider a European call option with maturity T and strike price K. A straightforward multiperiod extension of (1) gives its time t price as:

$$\pi_t = E_t \left[m_{t+1} m_{t+2} \cdots m_T \ (S_T - K)^+ \right]. \tag{10}$$

Garcia, Luger, and Renault (2001) provide a convenient set of general assumptions about the bivariate process $\left(m_{t+1}, \dfrac{S_{t+1}}{S_t} \right)$, which allows them to give closed-form formulas for the expectations (10) while encompassing the most usual options-pricing models (see also Garcia, Ghysels, and Renault 2003). The maintained assumptions are as follows:

Assumption A1

The variables $\left(m_{\tau+1}, \frac{S_{\tau+1}}{S_\tau} \right)_{1 \le \tau \le T-1}$ *are conditionally independent given the path,* $U_1^T = (U_t)_{1 \le t \le T-1}$, *of a vector,* U_t, *of state variables.*

Assumption A1 expresses that the dynamics of the returns is driven by the state variables. A similar assumption is made in common stochastic volatility models (the stochastic volatility process being the state variable) when standardized returns are assumed to be independent.

Assumption A2

The process $\left(m_t, \frac{S_t}{S_{t-1}} \right)$ *does not Granger-cause the state-variables process* (U_t).

This assumption holds that the state variables are exogenous. For common stochastic volatility or hidden Markov processes, such an exogeneity assumption is usually maintained to make the standard filtering strategies valid. It should be noted that this exogeneity assumption does not preclude instantaneous causality relationships, such as a leverage effect.

Assumption A3 *The conditional probability distribution of* $\left(\log m_{t+1}, \log \frac{S_{+1}}{S_t} \right)$, *given* U_1^{t+1}, *is a bivariate normal:*

$$\begin{bmatrix} \log\ m_{t+1} \\ \log \dfrac{S_{t+1}}{S_t} \end{bmatrix} | U_1^{t+1} \rightsquigarrow \mathcal{N} \left[\begin{pmatrix} \mu_{mt} \\ \mu_{st} \end{pmatrix}, \begin{pmatrix} \sigma_{mt}^2 & \sigma_{mst} \\ \sigma_{mst} & \sigma_{st}^2 \end{pmatrix} \right].$$

Assumption A3 is a very general version of the mixture-of-normals model. A maintained assumption is that investors observe U_t at time t, so that the conditioning information in the expectation operator (10) is:

$$I_t = \sigma \left[m_\tau, S_\tau, U_\tau, \tau \le t \right]. \tag{11}$$

In our simulation exercise, the mixing variable, U_{t+1}, will be a two-state Markov chain with a transition matrix:

$$P = \begin{bmatrix} p_{00} & 1-p_{00} \\ 1-p_{11} & p_{11} \end{bmatrix}. \tag{12}$$

Indeed, following Garcia, Luger, and Renault (2001), a general options-pricing formula can be stated for any Markov process (U_t) conformable to A1, A2, and A3.

Proposition 3.1 *Under assumptions A1, A2, and A3,*

$$\frac{\pi_t}{S_t} = \pi_t(x_t) = E_t\left\{Q_{ms}(t,T)\,\Phi\left(d_1(x_t)\right) - \frac{\tilde{B}(t,T)}{B(t,T)}e^{-x_t}\Phi\left(d_2(x_t)\right)\right\},$$

where $x_t = \log\frac{S_t}{KB(t,T)}$, $B(t,T) = E_t\left(\prod_{\tau=t}^{T-1} m_{\tau+1}\right)$ *is the time t price of a bond maturing at time T, and*

$$\begin{aligned} d_1(x) &= \frac{x}{\bar{\sigma}_{t,T}} + \frac{\bar{\sigma}_{t,T}}{2} + \frac{1}{\bar{\sigma}_{t,T}}\log\left[Q_{ms}(t,T)\frac{B(t,T)}{\tilde{B}(t,T)}\right], \\ d_2(x) &= d_1(x) - \bar{\sigma}_{t,T}, \\ \bar{\sigma}^2_{t,T} &= \sum_{\tau=t}^{T-1}\sigma^2_{s\tau}, \end{aligned}$$

and

$$\begin{aligned} \tilde{B}(t,T) &= \exp\left(\sum_{\tau=t}^{T-1}\mu_{m\tau+1} + \frac{1}{2}\sum_{\tau=t}^{T-1}\sigma^2_{m\tau},\right), \\ Q_{ms}(t,T) &= \tilde{B}(t,T)\exp\left(\sum_{\tau=t}^{T-1}\sigma_{ms\tau+1}\right)E\left[\frac{S_T}{S_t}|U_1^T\right]. \end{aligned}$$

As explicitly analyzed in Garcia, Ghysels, and Renault (2003), this general options-pricing formula encompasses most of the common pricing formulas for European options on equity.

To consider economically meaningful regime shifts in the SDF, it is convenient to start from a two-factor model as produced by Epstein and Zin (1989). Their recursive utility framework leads them to the following SDF:

$$m_{t+1} = \left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\right]^{\gamma}\left[\frac{1}{R_{mt+1}}\right]^{1-\gamma}, \tag{13}$$

where $\rho = 1 - \dfrac{1}{\sigma}$, σ is the elasticity of intertemporal substitution and $\gamma = \dfrac{\alpha}{\rho}$ with $(1-\alpha)$ the index of relative risk aversion. With a two-state mixing variable, U_{t+1}, $\log m_{t+1}$ appears as a mixture of two normal distributions in two cases. In the first case of state-dependent preferences, preference parameters are functions of U_{t+1}; in the second case, there are regime shifts in fundamentals and the joint probability distribution of $\left(\log \frac{C_{t+1}}{C_t}, \log R_{mt+1}\right)$ is a mixture of normals.

The case of state-dependent preferences has been analyzed recently by Melino and Yang (2003); Garcia, Luger, and Renault (2001, 2003) focus on shifts in fundamentals.[2]

3.2 State-dependent preferences or fundamentals

Let us first assume, as Melino and Yang (2003) do, that the three preference parameters β, α, ρ are all state-dependent and denoted as $\beta(U_t), \alpha(U_t)$, and $\rho(U_t)$. While these values, known by the investor at time t, define the investor's time t utility level, the investor does not know at this date the next values $\beta(U_{t+1}), \alpha(U_{t+1})$, and $\rho(U_{t+1})$. Therefore, the resulting SDF will be more complicated than just replacing α, β, and ρ in (13) by their state-dependent value. Melino and Yang (2003) show that the SDF is:

$$m_{t+1} = \left[\beta(U_t)\left(\frac{C_{t+1}}{C_t}\right)^{\rho(U_t) - \frac{\rho(U_t)}{\rho(U_{t+1})}}\right]^{\gamma(U_t)} R_{mt+1}^{\frac{\alpha(U_t)}{\rho(U_{t+1})} - 1} P_t^{\frac{\alpha(U_t)}{\rho(U_{t+1})} - \gamma(U_t)}, \tag{14}$$

where $\gamma(U_t) = \frac{\alpha(U_t)}{\rho(U_t)}$ and P_t is the time t price of the market portfolio. When $\beta(U_t)$, $\alpha(U_t)$, $\rho(U_t) = \rho(U_{t+1})$ are constants, this pricing kernel reduces to the Epstein and Zin SDF (13). By definition:

$$R_{mt+1} = \frac{P_{t+1} + C_{t+1}}{P_t},$$

while the underlying asset return is $\frac{S_{t+1} + D_{t+1}}{S_t}$. Asset prices P_t and S_t are then determined as discounted values of future dividend flows by iteration of the following pricing formulas:

$$P_t = E_t\left[m_{t+1}(P_{t+1} + C_{t+1})\right] \text{ and } S_t = E_t\left[m_{t+1}(S_{t+1} + D_{t+1})\right]. \tag{15}$$

Garcia, Luger, and Renault (2001) show that assumptions A1 and A2 are implied by similar assumptions stated for the joint process $\left(\frac{C_{t+1}}{C_t}, \frac{D_{t+1}}{D_t}\right)$. Assumption A3 will then also be implied by a similar assumption about fundamentals.

Assumption A3': *The conditional probability distribution of* $\left(\log \frac{C_{t+1}}{C_t}, \log \frac{D_{+1}}{D_t}\right)$, *given* U_1^{t+1}, *is a bivariate normal*

$$\begin{bmatrix} \log \frac{C_{t+1}}{C_t} \\ \log \frac{D_{+1}}{D_t} \end{bmatrix} | U_1^{t+1} \rightsquigarrow N\left[\begin{pmatrix} \mu_{X_{t+1}} \\ \mu_{Y_{t+1}} \end{pmatrix}, \begin{pmatrix} \sigma^2_{X_{t+1}} & \sigma_{XY,t+1} \\ \sigma_{XY,t+1} & \sigma^2_{Y_{t+1}} \end{pmatrix}\right].$$

[2]See also Gordon and St-Amour (2000) for an alternative way to introduce state dependence in preferences in a consumption capital asset-pricing model (CCAPM) framework.

Proposition 3.2 nests the results of Melino and Yang (2003) and Garcia, Luger, and Renault (2001) in a common setting.

Proposition 3.2 *Under assumptions A1, A2, and A3', with* m_{t+1} *given by (14), the conditional probability distribution of* $\left(\log m_{t+1}, \log \frac{S_{t+1}}{S_t}\right)$*, given* U_1^{t+1}*, is jointly normal, with mean and variances defined in the appendix.*

In the simulation exercises that we conduct in section 4, we will consider first regime changes in fundamentals and then regime changes in several configurations of the preference parameters, to disentangle the respective roles of fundamentals and preferences. The general options-pricing formula, which can also accommodate the case where both fundamentals and preferences change with the regime, is given in proposition 3.3.

First, it is worth noting that the equilibrium model characterizes the asset prices P_t and S_t as:

$$\begin{aligned} \frac{P_t}{C_t} &= \lambda\left(U_1^t\right) = E_t\left[m_{t+1}\left(1+\lambda\left(U_1^{t+1}\right)\right)\frac{C_{t+1}}{C_t}\right]. \\ \frac{S_t}{D_t} &= \varphi\left(U_1^t\right) = E_t\left[m_{t+1}\left(1+\varphi\left(U_1^{t+1}\right)\right)\frac{D_{t+1}}{D_t}\right]. \end{aligned}$$

Proposition 3.3, therefore, summarizes the options-pricing implications of propositions 3.1 and 3.2 in the simplest case of a unit time to maturity (T=t+1):

Proposition 3.3 *Under A1, A2, and A3', the European options price is given by:*

$$\pi_t = E_t\left[S_t Q_{XY}(t,t+1)\,\Phi(d_1) - K\widetilde{B}(t,t+1)\,\Phi(d_2)\right],$$

where,

$$d_1 = \frac{Log\left[\frac{S_t Q_{XY}(t,t+1)}{K\widetilde{B}(t,t+1)}\right]}{\left(\sigma_{Yt+1}^2\right)^{\frac{1}{2}}} + \frac{1}{2}\left(\sigma_{Yt+1}^2\right)^{\frac{1}{2}},\ d_2 = d_1 - \left(\sigma_{Yt+1}^2\right)^{\frac{1}{2}},$$

with,

$$\widetilde{B}(t,t+1) = a\left(U_1^{t+1}\right)\exp\left(\left[\alpha\left(U_{t+1}\right)-1\right]\mu_{X_{t+1}} + \frac{1}{2}\left[\alpha\left(U_{t+1}\right)-1\right]^2\sigma_{X_{t+1}}^2\right),$$

$$Q_{XY}(t,t+1) = \widetilde{B}(t,t+1)\,b_t^{t+1}\frac{\varphi\left(U_1^t\right)}{\varphi\left(U_1^{t+1}\right)}\exp\left(\left[\alpha\left(U_{t+1}\right)-1\right]\rho_{XY}\sigma_{X\tau}\sigma_{Y\tau}\right)E_t\left[\frac{S_{t+1}}{S_t}|U_1^{t+1}\right],$$

and

$$a\left(U_1^{t+1}\right) = \beta\left(U_t\right)^{\frac{\alpha(U_t)}{\rho(U_t)}}\left(1-\beta\left(U_t\right)\right)^{\frac{\alpha(U_t)}{\rho(U_{t+1})}-\frac{\alpha(U_t)}{\rho(U_t)}}\lambda\left(U_1^t\right)^{1-\frac{\alpha(U_t)}{\rho(U_t)}}\left[1+\lambda\left(U_1^{t+1}\right)\right],$$

$$b_t^{t+1} = \frac{1+\varphi\left(U_1^{t+1}\right)}{\varphi\left(U_1^t\right)}.$$

PROOF. The proof is similar to the proof where $\alpha(U_t)$, $\beta(U_t)$, and $\rho(U_t)$ are constants, which is given in Garcia, Luger, and Renault (2003). ■

If the preference parameters α, β, and ρ are constants, proposition 3.3 collapses to the Garcia, Luger, and Renault (2001) options-pricing formula. Note that the definition of $\lambda\left(U_1^{t+1}\right)$ and $\varphi\left(U_1^{t+1}\right)$ is equivalent to: $E_t Q_{XY}(t, t+1) = 1$, and $E_t \widetilde{B}(t, t+1) = B(t, t+1)$.

3.3 Simulating options and stock prices

First, we calibrate our economic models with regime shifts in the parameters describing preferences or economic fundamentals.[3] We then use proposition 3.3 to compute options prices with different strike prices. To use the methodology described in (7), we need to develop a simple technique to create bid-ask spreads around the simulated prices. This is done in three steps:

- Step 1: Given the stock price, S_t, we find a bid-ask spread, *sp*, by drawing in a lognormal distribution:

$$\log(sp) \to N\left(\mu_{sp}, \sigma_{sp}^2\right),$$

  where the parameters μ_{sp} and σ_{sp}^2 are chosen exogenously.

- Step 2: Given *sp*, we draw a real number, x, in the censored normal probability distribution $N(\mu_x, \sigma_x^2)$, given $0 \leq x \leq sp$.

- Step 3: We then compute the stock bid and ask prices:

$$\begin{aligned} \textit{ask price} &= S_t + (sp - e^x), \\ \textit{bid price} &= S_t - e^x. \end{aligned}$$

We apply a similar simulation methodology to create bid and ask prices for options. Based on these bid and ask options prices and stock prices, we recover the risk-neutral probabilities using the non-parametric methodology described in section 2. It is important to note that our Monte Carlo approach gives us the historical return distribution. Therefore, we do not need to use any non-parametric estimation technique to recover the historical distribution.

The whole procedure must be applied for each state $U_t \in \{0, 1\}$ of the economy. At date t, given the state variable value, $U_t \in \{0, 1\}$, we compute the call option prices:

$$\pi_t(U_t) = E\left[S_t Q_{XY}(t, t+1)\,\Phi(d_1) - K\widetilde{B}(t, t+1)\,\Phi(d_2)\,|U_t\right],$$

[3] In the case of state-dependent fundamentals, we choose values that are close to those estimated in Garcia, Luger, and Renault (2003), where preference parameters are not state-dependent. For state-dependent preferences, we disturb these particular values. All values used are explicitly given in Figures 1 to 6.

and perform steps 1, 2, and 3. We then use equations (4) and (5) to infer the conditional ARA and pricing-kernel functions across states (given the state variable, U_t).

By construction, these quantities are computed from probabilities $p_j(U_t)$ and $p_j^*(U_t)$, which explicitly depend on the actual value of the latent state, U_t. By contrast, a statistician who does not observe the state and performs a non-parametric estimation of the stationary historical distribution that does not account for unobserved heterogeneity, will estimate marginal probabilities, p_j, that are averaged across states:

$$p_j = P(U_t = 0) p_j(0) + P(U_t = 1) p_j(1). \tag{16}$$

As far as risk-neutral probability, p_j^*, is concerned, the issue is less clear. If we could be sure that not only the agents have observed the state U_t but also that the statistical observation of asset prices is synchronized with observations, then the p_j^* computed from (6) and the real data should be $p_j^*(U_t)$. Any synchronization problem, however, may push the implied p_j^* towards their averaged values:

$$p_j^* = P(U_t = 0) p_j^*(0) + P(U_t = 1) p_j^*(1). \tag{17}$$

For reasons made explicit below, we choose to compare the implied risk aversion and pricing kernel computed state-by-state from $\left(p_j(U_t), p_j^*(U_t)\right)$ with the fully marginalized ones; that is, computed from marginalized values $\left(p_j, p_j^*\right)$ given by (16) and (17), rather than using the possible mixed approach $\left(p_j, p_j^*(U_t)\right)$.

4. Empirical Results

4.1 Choosing preference and fundamental parameters

In the case of state-dependent fundamentals, we choose values that are close to those estimated in Garcia, Luger, and Renault (2003), where preference parameters are not state-dependent. Garcia, Luger, and Renault propose a utility-based options-pricing model with stochastic volatility and jumps features. Their model is cast within the recursive utility framework of Epstein and Zin (1989), in which the roles of discounting risk aversion and intertemporal substitution are disentangled. Garcia, Luger, and Renault use daily price data for S&P 500 index European call options obtained from the Chicago Board Options Exchange for the period January 1991 to December 1995. They also use daily return data for the S&P 500 index and estimate preferences parameters in S&P 500 options prices. They find quite reasonable values for the coefficient of the risk aversion and the intertemporal elasticity of substitution. Depending on the sample period used, estimates are in the range of (-8.75,-4) (see Table 5 in their paper). Garcia, Luger, and Renault's estimation approach is based on the method of moments. To investigate whether their approach produces a good estimate, they first calibrate their model, and then simulate options prices and estimate the preference parameters holding the fundamental parameters fixed. Depending on the approach used to simulate data, estimates of α are in the range of (-3.5,-2.46) and estimates of ρ are in the range of (-11.07,-10.57). Garcia, Luger, and Renault find that the estimates based on stock returns are more biased than the estimators based on moment conditions for options. The main conclusion of their estimation approach is that it produces quite good estimates with a panel of options prices rather than a time series on the underlying asset. For our results, based on state-dependent preferences described below, we disturb preference parameters values around Garcia, Luger, and Renault's results but still maintain them in the realistic range. To illustrate our economic models and their effects on the puzzles put forward by Jackwerth (2000), we start by analyzing the cases of state dependence in fundamentals and in preferences separately, and then we allow for regime switching in both fundamentals and preferences simultaneously.

4.2 Regime shifts in fundamentals

We first assume that only the fundamentals are affected by the latent state variables. Based on the prices generated with the procedures described in the previous sections, we follow the methodology described in section 2 and recover the risk-aversion and pricing-kernel functions across wealth states. As stated earlier, these implied values are obtained with the goodness-of-fit criterion (7). The left graph in Figure 1 reveals that the unconditional pricing kernel increases in the centre wealth states (over the range of 0.9 to 1.1). This feature is highlighted in Brown and Jackwerth (2000) as the pricing-kernel puzzle. We use the term unconditional to emphasize that the pricing-kernel function across wealth states is computed using marginalized probabilities given by (16) and (17). However, the pricing kernels in each regime decline monotonically across wealth states. In the right panel of Figure 1, we plot the unconditional ARA and regime-dependent functions. For the centre wealth states, we find, as in Jackwerth (2000), that the risk-aversion function becomes negative. Within each regime, the ARA functions across wealth states are perfectly decreasing functions of the aggregate wealth: the puzzles disappear. In Figure 2, we compare the pricing-kernel and the ARA functions obtained using the goodness-of-fit criterion with the ones derived using the Hansen-Jagannathan distance in (8). The two panels confirm that the results do not depend on the particular distance measure used. The same features are exhibited with the alternative Hansen and Jagannathan (1997) distance measure.

4.3 Regime shifts in preferences

We next consider state dependence in the investor's preference parameters and investigate several state-dependent preference cases. First, we assume a constant relative risk aversion (CRRA) and a state-dependent elasticity of intertemporal substitution (EIS). Second, we assume a state-dependent risk aversion and a constant EIS. Third, we assume cyclical CRRA and EIS, and, finally, we assume state-dependent time preferences. The state-dependent preference parameters are obtained by disturbing the preference parameters used in Garcia, Luger, and Renault (2003). As a number of papers show, plausible relative risk-aversion parameters lie between 0 and 10 (see, for example, Garcia, Luger, and Renault 2003). Therefore, we keep the disturbed values in the same range. This makes the state-dependent risk-aversion parameters reasonable in each regime. For each of the three types of state-dependency described above, we get very similar results: both the unconditional pricing-kernel and ARA functions exhibit the aforementioned puzzles, whereas the puzzles disappear within each regime. Therefore, we report only the results for state-dependent relative risk aversion and constant EIS in Figure 3. Around the centre wealth states, we observe an increasing marginal utility in the left panel, while the risk aversion shown in the right panel falls into negative values. Figure 4 confirms these results with the alternative distance measure (8).

4.4 General comments

Regime shifts in fundamentals or in preferences, or in both, as illustrated in Figures 5 and 6, lead us to the same general conclusion. While the implied risk aversion and the implied pricing kernel computed from marginalized probabilities $\left(p_j, p_j^*\right)$ display the same paradoxical features as in Aït-Sahalia and Lo (2000) and Jackwerth (2000), taking into account unobserved heterogeneity through the state-dependent probabilities $\left(p_j\left(U_t\right), p_j^*\left(U_t\right)\right)$ solves the puzzle. In other words, our results lead us to think that possibly investors' utility functions are not at odds with traditional economic theory, but that investors observe a latent state variable that artificially creates a paradox when it is forgotten in the statistical procedure. As noted earlier, full observation of states by agents, in perfect synchronization with our observation of options prices, may suggest that the probabilities $\left(p_j, p_j^*\left(U_t\right)\right)$ should be used instead. The implied risk aversion and pricing kernel observed with such mixed probabilities appear, according to a complementary simulation study available upon request, even wilder than the ones produced by marginal probabilities $\left(p_j, p_j^*\right)$. Since the latter look more conformable to the empirical evidence put forward by Jackwerth (2000), we have chosen to focus on them in this paper.

5. Conclusion

In this paper, we have investigated the ability of economic models with regime shifts to produce and solve the risk-aversion and the pricing-kernel puzzles put forward by Aït-Sahalia and Lo (2000) and Jackwerth (2000). We have shown that models with regime shifts in fundamentals or an investor's preferences can explain and rationalize these puzzles. The ARA and pricing-kernel functions extracted from the simulated prices in these economies exhibit the same puzzling features as in papers by previous researchers, and are inconsistent with the usual assumptions of decreasing marginal utility and positive risk aversion. Within each regime, however, the ARA and pricing-kernel functions are consistent with economic theory: the investor's utility is concave and their risk aversion remains positive. In other words, investor behaviour is not at odds with economic theory, but depends on some factors that the statistician does not observe. We have shown that this conclusion is robust to the choice of the statistical estimation procedure.

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Figure 1: **Pricing-kernel (PK) and absolute risk-aversion (ARA) functions with state dependence in fundamentals**. The preference parameters are: $\beta = 0.95$, $\alpha = -5$, $\rho = -11$. The regime probabilities are: $p_{11} = 0.9$, $p_{00} = 0.6$. For the economic fundamentals, the means of the consumption growth rate are $\mu_{X_{t+1}} = (0.0015, -0.0009)$ and the corresponding standard deviations $\sigma_{X_{t+1}} = (0.0159, 0.0341)$. For the dividend rate, the parameters are $\mu_{Y_{t+1}} = (0, 0)$, $\sigma_{Y_{t+1}} = (0.02, 0.12)$. The correlation coefficient between consumption and dividends is 0.6. The number of options used is 50. The number of wealth states is $n = 170$. The left-hand panel contains the conditional and unconditional PK functions across wealth states. The right-hand panel contains the conditional and unconditional ARA functions across wealth states. The conditional ARA (PK) function is the ARA (PK) function computed within each regime. The unconditional ARA (PK) function is the ARA (PK) function computed when regimes are not observed.

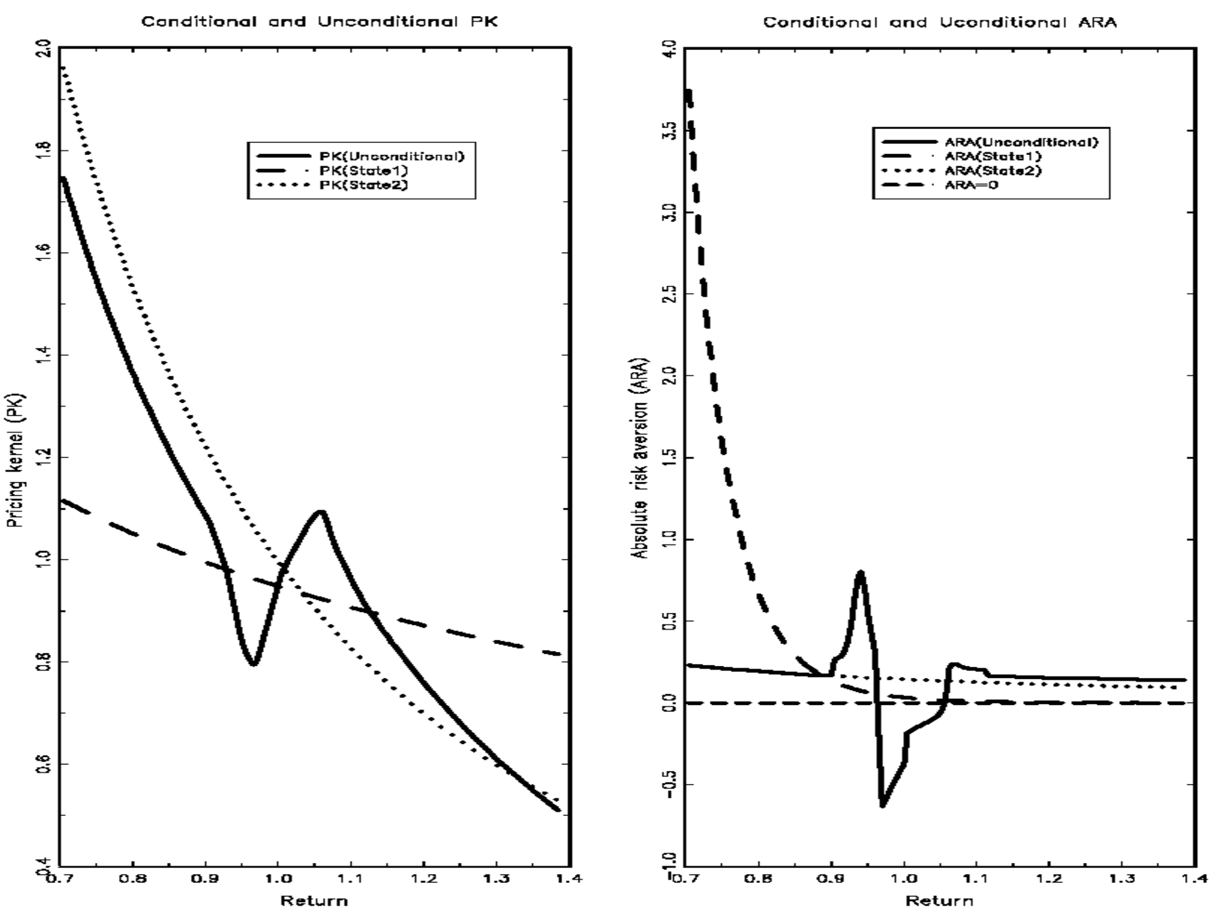


Figure 2: **Comparison of pricing-kernel (PK) and absolute risk-aversion (ARA) functions with state dependence in fundamentals for two distance measures:** The preference parameters are: $\beta = 0.95$, $\alpha = -5$, $\rho = -11$. The regime probabilities are: $p_{11} = 0.9$, $p_{00} = 0.6$. For the economic fundamentals, the means of the consumption growth rate are $\mu_{X_{t+1}} = (0.0015, -0.0009)$, and the corresponding standard deviations $\sigma_{X_{t+1}} = (0.0159, 0.0341)$. For the dividend rate, the parameters are $\mu_{Y_{t+1}} = (0, 0)$, $\sigma_{Y_{t+1}} = (0.02, 0.12)$. The correlation coefficient between consumption and dividends is 0.6. The number of options used is 50. The number of wealth states is $n = 170$. The left-hand panel contains the unconditional PK function across wealth states for the goodness-of-fit and the Hansen and Jagannathan (1997) distance measures. The right-hand panel contains the unconditional ARA function across wealth states for the goodness-of-fit and the Hansen and Jagannathan (1997) distance measures. The unconditional ARA (PK) function is the ARA (PK) function computed when regimes are not observed.

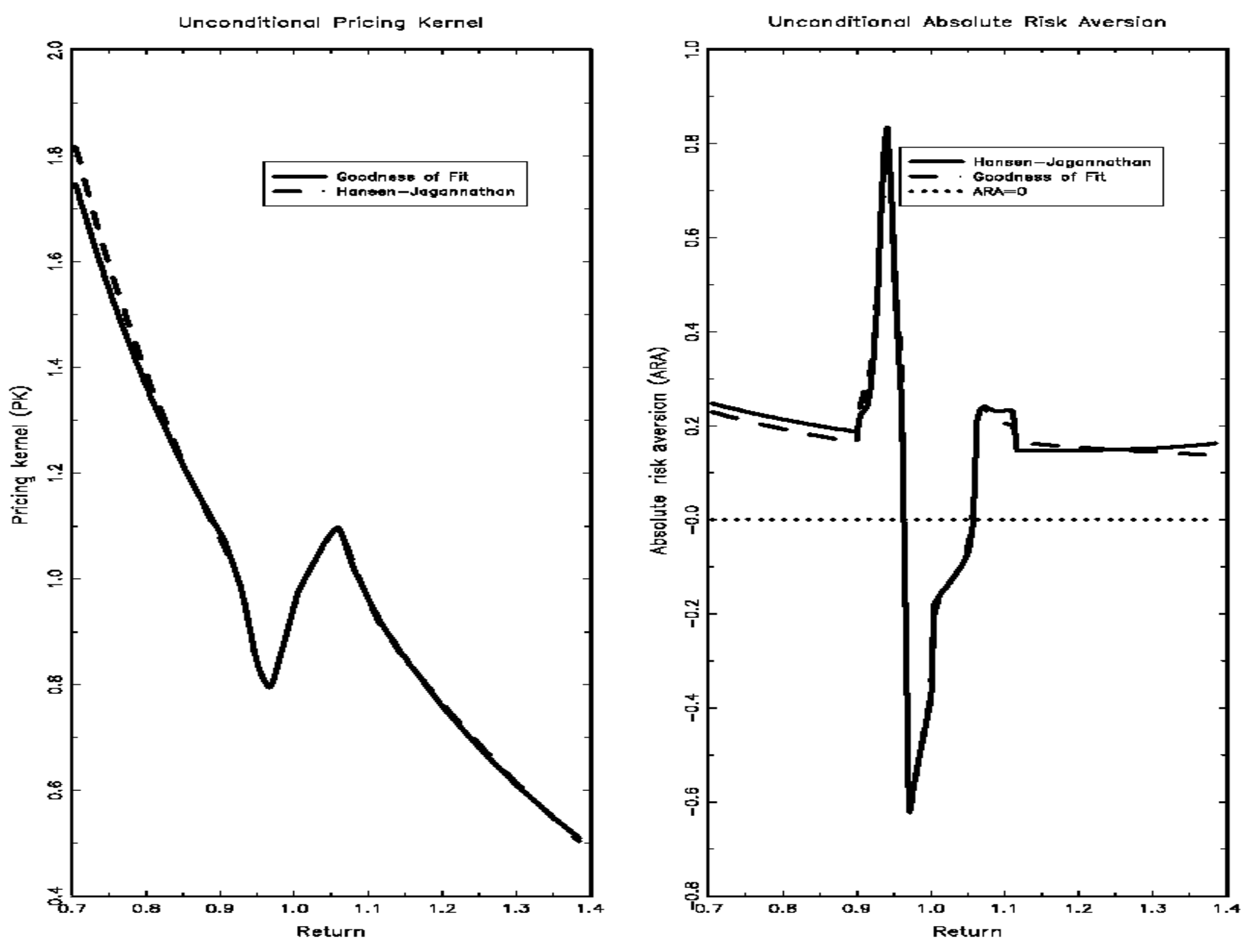


Figure 3: **Pricing-kernel (PK) and absolute risk-aversion (ARA) functions with state dependence in preferences**. The preference parameters are $\beta = 0.95$, $\alpha = (-7, -4.8)$, $\rho = -10$. The regime probabilities are $p_{11} = 0.9$, $p_{00} = 0.6$. For the economic fundamentals, the mean of the consumption growth rate is $\mu_{X_{t+1}} = 0.018$ and the standard deviation $\sigma_{X_{t+1}} = 0.037$. For the dividend rate, Y_{t+1}, the parameters are $\mu_{Y_{t+1}} = -0.0018$, $\sigma_{Y_{t+1}} = 0.12$. The correlation coefficient between consumption and dividend is 0.6. The number of options used is 50. The number of wealth states is $n = 170$. The left-hand panel contains the conditional and unconditional PK functions across wealth states. The right-hand panel contains the conditional and unconditional ARA functions across wealth states. The conditional ARA (PK) function is the ARA (PK) function computed within each regime. The unconditional ARA (PK) function is the ARA (PK) function computed when regimes are not observed.

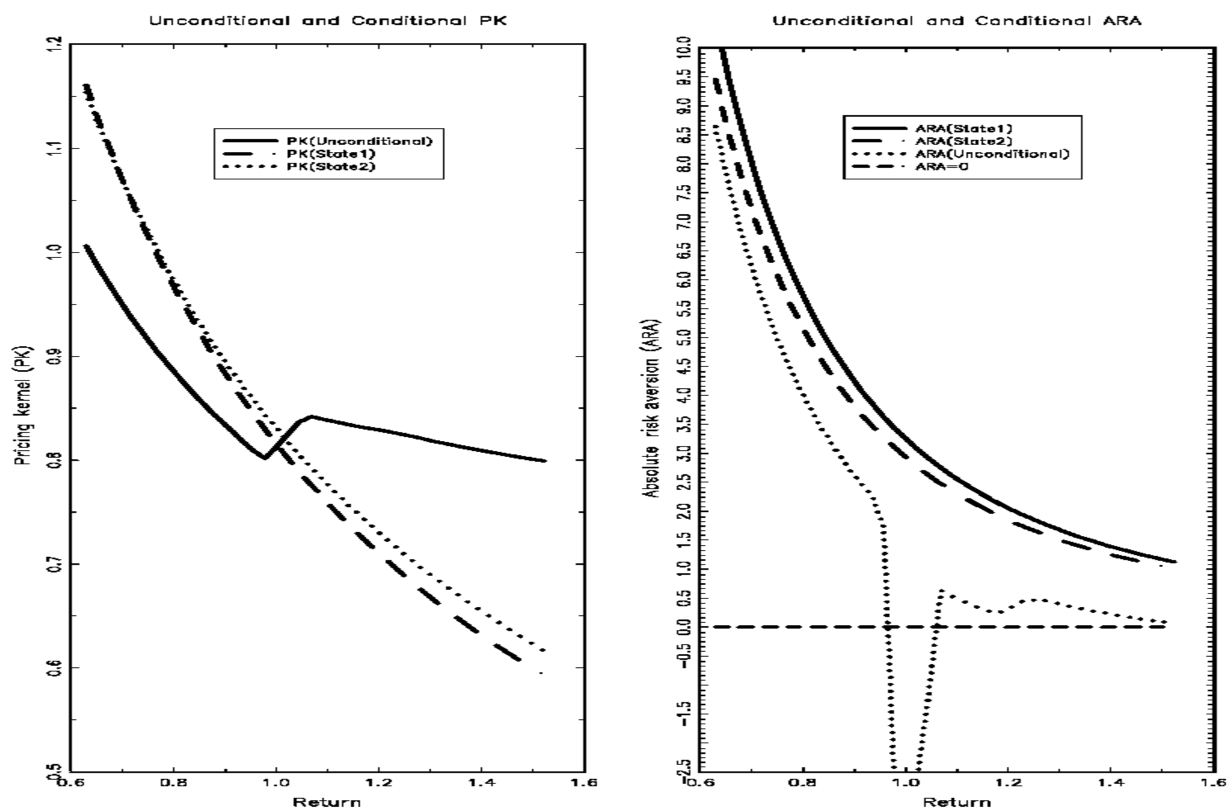


Figure 4: **Comparison of pricing-kernel (PK) and absolute risk-aversion (ARA) functions with state dependence in preferences**. The preference parameters are $\beta = 0.97$, $\alpha = (-7, -4.8)$, $\rho = -10$. The regime probabilities are $p_{11} = 0.9$, $p_{00} = 0.6$. For the economic fundamentals, the mean of the consumption growth rate is $\mu_{X_{t+1}} = 0.018$ and the standard deviation $\sigma_{X_{t+1}} = 0.037$. For the dividend rate, Y_{t+1}, the parameters are $\mu_{Y_{t+1}} = -0.0018$, $\sigma_{Y_{t+1}} = 0.12$. The correlation coefficient between consumption and dividend is 0.6. The number of options used is 50. The number of wealth states is $n = 170$. The left-hand panel contains the unconditional ARA function across wealth states for the goodness-of-fit and the Hansen and Jagannathan (1997) distance measures. The right-hand panel contains the unconditional ARA function across wealth states for the goodness-of-fit and the Hansen and Jagannathan (1997) distance measures. The unconditional ARA (PK) function is the ARA (PK) function computed when regimes are not observed.

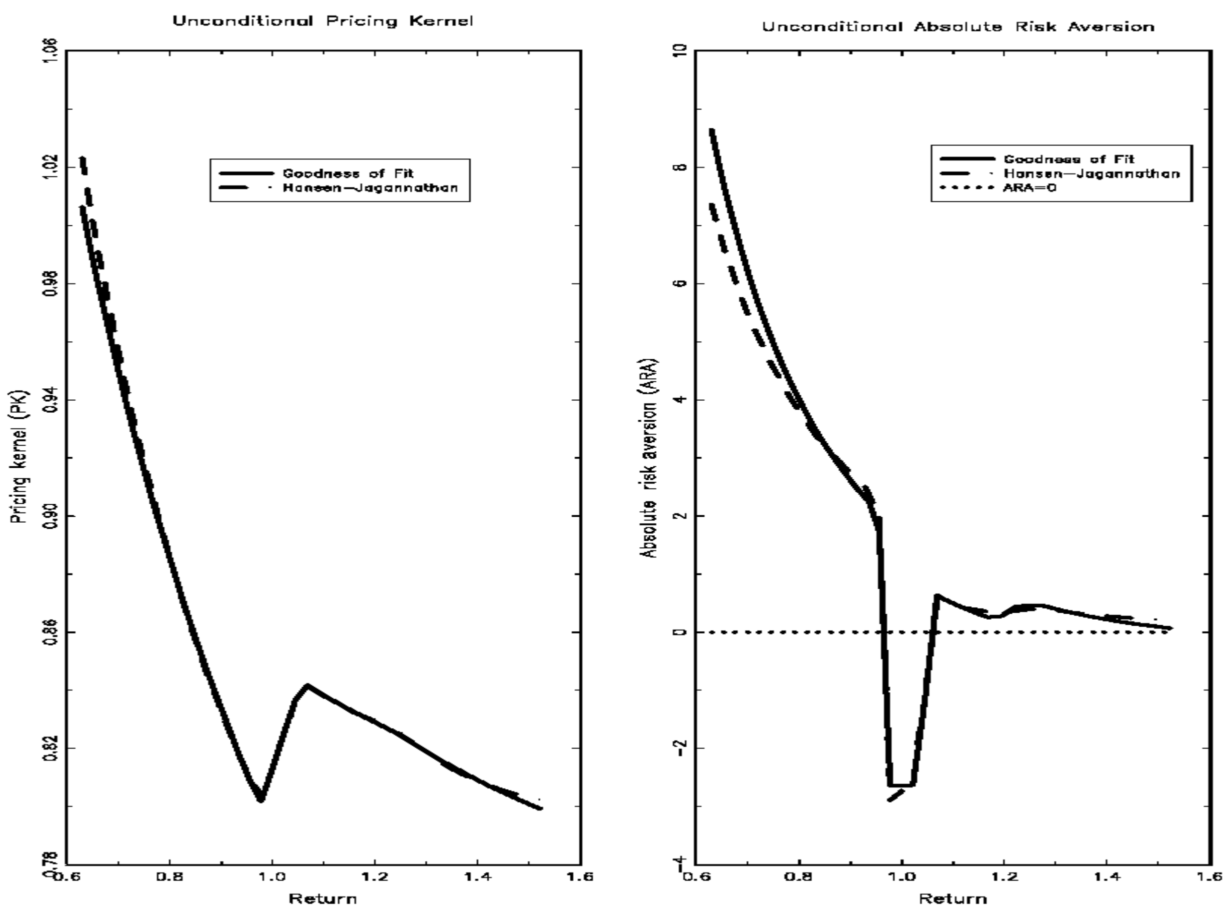
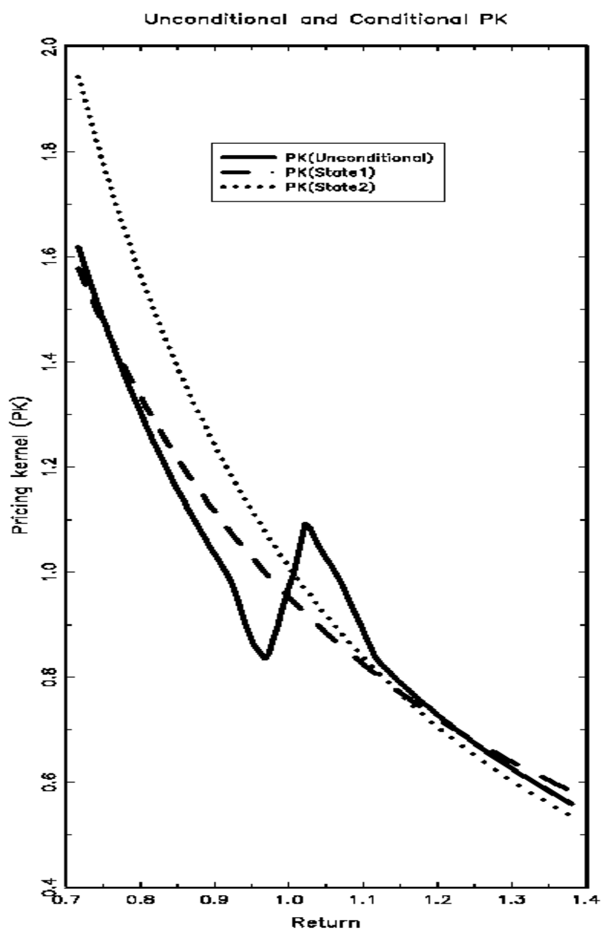


Figure 5: **Pricing-kernel (PK) and absolute risk-aversion (ARA) functions with state dependence in both preferences and fundamentals**. The preference parameters are $\beta = 0.95$, $\alpha = (-5, -3.5)$, $\rho = -10$. The regime probabilities are $p_{11} = 0.9$, $p_{00} = 0.6$. For the economic fundamentals, the means of the consumption growth rate are $\mu_{X_{t+1}} = (0.0015, -0.0009)$ and the standard deviations $\sigma_{X_{t+1}} = (0.0159, 0.0341)$. For the dividend rate, Y_{t+1}, the parameters are $\mu_{Y_{t+1}} = (0, 0)$, $\sigma_{Y_{t+1}} = (0.02, 0.12)$. The correlation coefficient between consumption and dividend is 0.6. The number of options used is 50. The number of wealth states is $n = 170$. The left-hand panel contains the conditional and unconditional PK functions across wealth states. The right-hand panel contains the conditional and unconditional ARA functions across wealth states. The conditional ARA (PK) function is the ARA (PK) function computed within each regime. The unconditional ARA (PK) function is the ARA (PK) function computed when regimes are not observed.



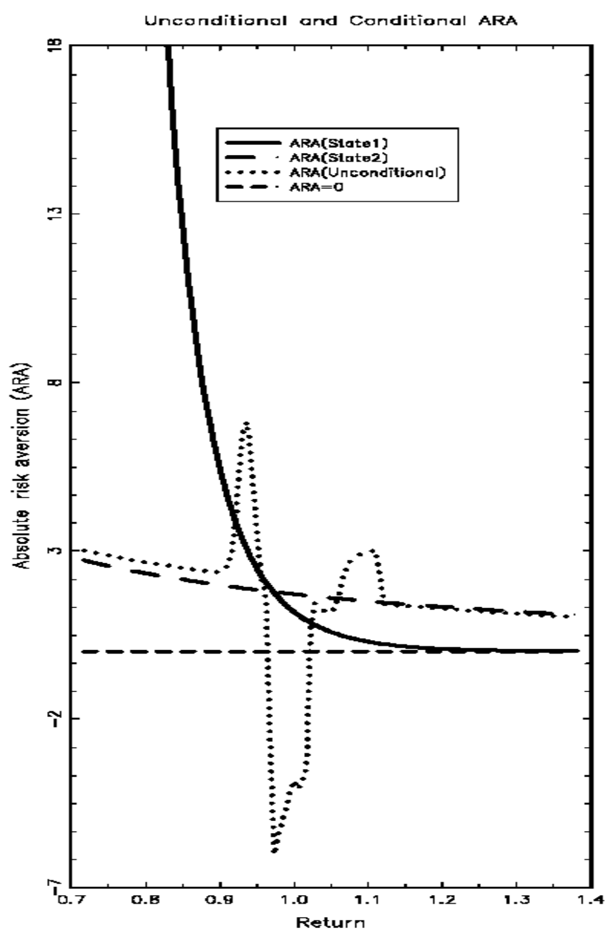
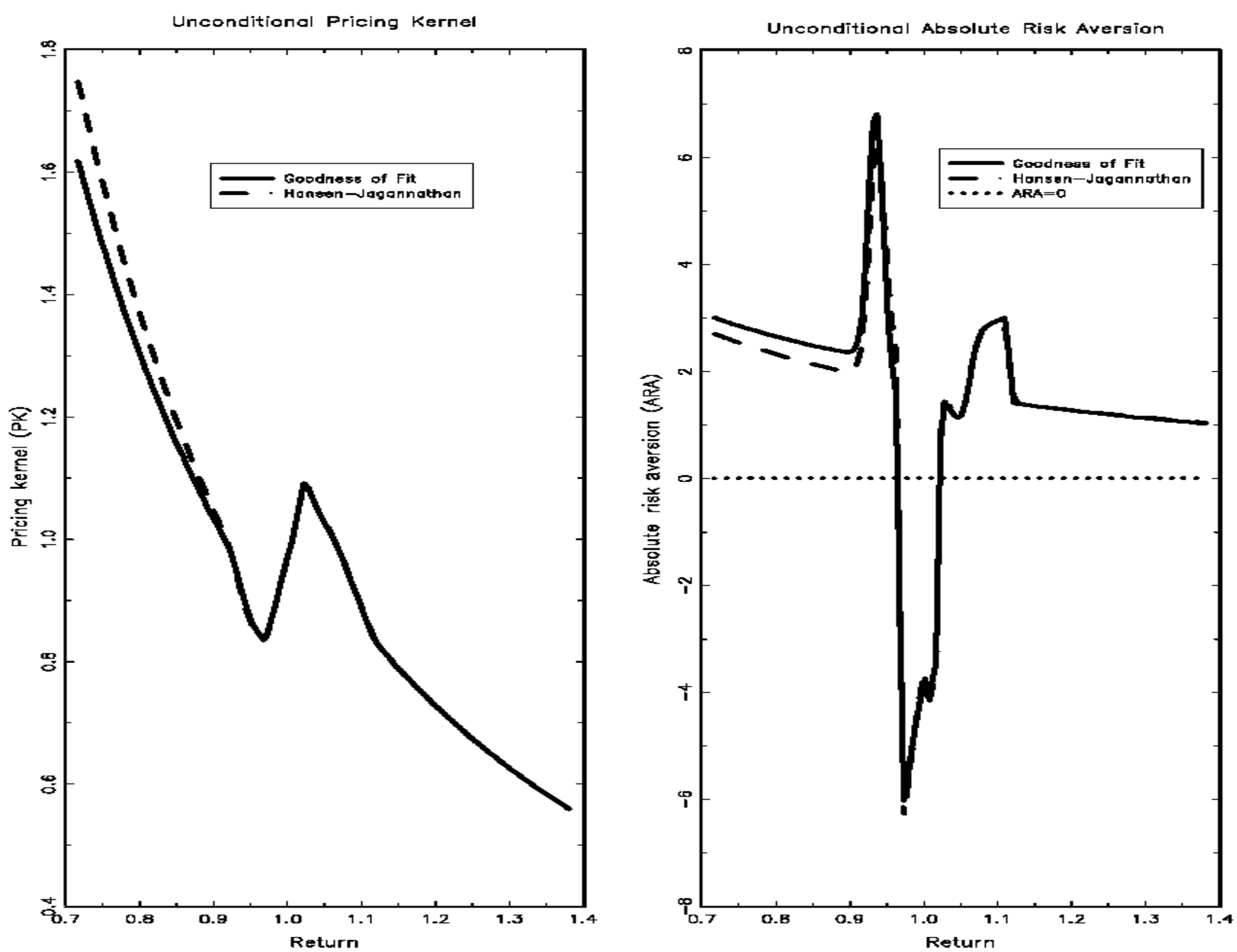


Figure 6: **Comparison of pricing-kernel (PK) and absolute risk-aversion (ARA) functions with state dependence in both preferences and fundamentals**. The preference parameters are $\beta = 0.95$, $\alpha = (-5, -3.5)$, $\rho = -10$. The regime probabilities are $p_{11} = 0.9$, $p_{00} = 0.6$. For the economic fundamentals, the means of the consumption growth rate are $\mu_{X_{t+1}} = (0.0015, -0.0009)$ and the standard deviations $\sigma_{X_{t+1}} = (0.0159, 0.0341)$. For the dividend rate, Y_{t+1}, the parameters are $\mu_{Y_{t+1}} = (0, 0)$, $\sigma_{Y_{t+1}} = (0.02, 0.12)$. The correlation coefficient between consumption and dividend is 0.6. The number of options used is 50. The number of wealth states is $n = 170$. The left-hand panel contains the unconditional PK function across wealth states for the goodness-of-fit and Hansen and Jagannathan (1997) distance measures. The right-hand panel contains the unconditional ARA function across wealth states for the goodness-of-fit and Hansen and Jagannathan (1997) distance measures. The unconditional ARA (PK) function is the ARA (PK) function computed when regimes are not observed.



Appendix: Proof of Proposition 3.2

PROOF OF PROPOSITION 3.2. Rearranging equation (6.9) for the pricing kernel in Melino and Yang (2003), one obtains:

$$m_{t+1} = \left[\beta(U_t)\left(\frac{C_{t+1}}{C_t}\right)^{\rho(U_t)-\frac{\rho(U_t)}{\rho(U_{t+1})}}\right]^{\gamma(U_t)} R_{mt+1}^{\frac{\alpha(U_t)}{\rho(U_{t+1})}-1} P_t^{\frac{\alpha(U_t)}{\rho(U_{t+1})}-\frac{\alpha(U_t)}{\rho(U_t)}},$$

where $\gamma(U_t) = \frac{\alpha(U_t)}{\rho(U_t)}$ and P_t is the equilibrium price of the market portfolio at time t. If $\rho(U_t) = \rho(U_{t+1})$ and $\beta(U_t)$, $\alpha(U_t)$, $\rho(U_t)$ are constants, this pricing kernel reduces to the Epstein and Zin (1989) pricing kernel. Let $\varphi(U_t) = \frac{S_t}{D_t}$ denote the price-dividend ratio and $\lambda(U_t) = \frac{P_t}{C_t}$ the price-earnings ratio. The return on the market portfolio can be written as

$$R_{mt+1} = \frac{P_{t+1} + C_{t+1}}{P_t} = \left(\frac{\lambda\left(U_1^{t+1}\right)+1}{\lambda(U_t)}\right)\left(\frac{C_{t+1}}{C_t}\right),$$

and the stock return:

$$\frac{S_{t+1}}{S_t} = \frac{\varphi\left(U_1^{t+1}\right)}{\varphi(U_t)}\frac{D_{t+1}}{D_t}.$$

Let us assume that the conditional probability distribution of $\left(\log\frac{C_{t+1}}{C_t}, \log\frac{D_{t+1}}{D_t}\right)$, given U_1^{t+1}, is a bivariate normal:

$$\left[\begin{array}{c}\log\frac{C_{t+1}}{C_t}\\ \log\frac{D_{t+1}}{D_t}\end{array}\right] / U_1^T \rightsquigarrow N\left[\left(\begin{array}{c}\mu_{X_{t+1}}\\ \mu_{Y_{t+1}}\end{array}\right), \left(\begin{array}{cc}\sigma^2_{X_{t+1}} & \sigma_{XY,t+1}\\ \sigma_{XY,t+1} & \sigma^2_{Y_{t+1}}\end{array}\right)\right], \tag{A.1}$$

with $U_1^{t+1} = (U_\tau)_{1\leq\tau\leq t+1}$. Taking the logarithm of m_{t+1}, we get

$$\begin{aligned}\log m_{t+1} \quad = \quad & \gamma(U_t)\log\beta(U_t) + \left(\frac{\alpha(U_t)}{\rho(U_{t+1})} - 1\right)\log\left(\frac{\lambda\left(U_1^{t+1}\right)+1}{\lambda(U_t)}\right) + \left(\frac{\alpha(U_t)}{\rho(U_{t+1})} - \frac{\alpha(U_t)}{\rho(U_t)}\right)\log\left(\lambda(U_t)C_t\right) + \\ & \left[\gamma(U_t)\left(\rho(U_t) - \frac{\rho(U_t)}{\rho(U_{t+1})}\right) + \left(\frac{\alpha(U_t)}{\rho(U_{t+1})} - 1\right)\right]\log\left(\frac{C_{t+1}}{C_t}\right).\end{aligned}$$

The logarithm of the stock return is

$$\log\frac{S_{t+1}}{S_t} = \log\frac{\varphi\left(U_1^{t+1}\right)}{\varphi(U_t)} + \log\frac{D_{t+1}}{D_t}.$$

Consequently,

$$\left[\begin{array}{c}\log m_{t+1}\\ \log\frac{S_{t+1}}{S_t}\end{array}\right] = A + B\left[\begin{array}{c}\log\frac{C_{t+1}}{C_t}\\ \log\frac{D_{t+1}}{D_t}\end{array}\right],$$

where $A = (a_1, a_2)'$ with

$$\begin{aligned} a_1 &= \gamma(U_t)\log\beta(U_t) + \left(\rho(U_t) - \frac{\rho(U_t)}{\rho(U_{t+1})}\right)\log\left(\frac{\lambda\left(U_1^{t+1}\right)+1}{\lambda(U_t)}\right) + \left(\frac{\alpha(U_t)}{\rho(U_{t+1})} - \frac{\alpha(U_t)}{\rho(U_t)}\right)\log(\lambda(U_t)C_t), \\ a_2 &= \log\frac{\varphi\left(U_1^{t+1}\right)}{\varphi(U_t)}, \end{aligned}$$

and B is a diagonal matrix with diagonal coefficients:

$$\begin{aligned} b_{11} &= \left[\gamma(U_t)\left(\rho(U_t) - \frac{\rho(U_t)}{\rho(U_{t+1})}\right) + \left(\frac{\alpha(U_t)}{\rho(U_{t+1})} - 1\right)\right]. \\ b_{22} &= 1. \end{aligned}$$

Using (A.1), it is straightforward to show that:

$$\begin{bmatrix} \log m_{t+1} \\ \log\frac{S_{t+1}}{S_t} \end{bmatrix} / U_1^{t+1} \rightsquigarrow N[\mu, \Sigma_{ms}],$$

with

$$\begin{aligned} \mu &= A + B\begin{pmatrix} \mu_{X_{t+1}} \\ \mu_{Y_{t+1}} \end{pmatrix}, \\ \Sigma_{ms} &= B\begin{pmatrix} \sigma^2_{X_{t+1}} & \sigma_{X_{t+1}Y_{t+1}} \\ \sigma_{X_{t+1}Y_{t+1}} & \sigma^2_{Y_{t+1}} \end{pmatrix}B. \end{aligned}$$

This completes the proof. ■

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