Optimal Monetary Policy under Incomplete Markets and Aggregate Uncertainty: A Long-Run Perspective*

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August 2010

Abstract

This paper examines the role of monetary policy in an environment with aggregate risk and incomplete markets. In a two-period overlapping-generations model with aggregate uncertainty, optimal monetary policy attains the ex-ante Pareto optimal allocation. This policy aims to stabilize the savings rate in the economy by changing real returns of nominal bonds via variation in expected inflation. Optimal expected inflation is procylical and on average higher than without uncertainty. Simple inflation targeting rules closely approximate the optimal monetary policy.

JEL classifications: E5

Keywords: Optimal monetary policy; Inflation targeting.

*We thank Robert Amano, Steve Ambler, Pierre Duguay, John Murray, Peter Ireland, Larry Jones, Sharon Kozicki, seminar participants at the Bank of Canada, Higher School of Economics (Moscow), National Bank of Kazakhstan, Queen's University, and Université de Montréal for their comments and suggestions.

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1. Introduction

What is the role of monetary policy in an environment with aggregate risk and incomplete asset markets? We study a two-period overlapping-generations model (OLG) in which aggregate-income uncertainty and incomplete markets lead to suboptimal levels of savings and consumption. The ex-ante Pareto optimal allocation can be achieved through monetary policy. The optimal monetary policy stabilizes savings rates by affecting the expected real return on nominal bonds. It is characterized by: 1) expected inflation that on average is higher than without uncertainty, 2) a positive correlation between expected inflation and income, and 3) volatility of expected inflation that is inversely related to income persistence.

The characteristic properties of the optimal monetary policy stem from the tension between individually optimal savings decisions under incomplete markets, and the socially optimal allocation of consumption across generations. When faced with uninsurable income risk and a constant rate of return on savings, risk averse individuals smooth their consumption by varying their savings with income. When current income is higher than expected future income, individuals save more to move part of the current "windfall" into the future. When current income is lower than expected future income, individuals save less taking advantage of the anticipated increase in future income. In the presence of income heterogeneity across individuals, the lack of risk-sharing leads to savings rates that are more volatile and on average higher than those chosen by the social planner. When income is correlated across individuals, as in our model, due to aggregate shocks, the level of *aggregate* savings is not socially optimal.

We first analyze a tractable endowment economy where aggregate endowment shocks create ex-post income heterogeneity across households. Limited trading opportunities between generations restrict risk-sharing leading to suboptimally high variations in the savings rates of young households who are trying to self insure by varying their savings rates with income. With nominal assets being the only savings vehicle in this economy, the individual savings behavior of the young directly affects the allocation of goods between the young and the old because it determines the price of nominal assets sold by the old to the young. As a result of price level fluctuations, the young face uncertainty regarding the ex-post real rate of return on their nominal savings. The ex-post return on nominal assets depends on the realization of income of the young next period, and on inflation. Monetary policy can mitigate suboptimal fluctuations in savings rates by varying the expected inflation. In order to lower the average level and variability of savings rates, the optimal expected inflation is positive on average and procyclical. However, the degree to which expected inflation responds to income fluctuations depends on the persistence of the income disturbances. When income fluctuations are long-lived, individual incentives to vary savings are weak, which makes sizeable variations in the expected inflation unnecessary. Whereas when income movements are transitory, individuals have a strong incentive to vary their savings rates to smooth consumption across time. As a result, optimal inflation becomes more responsive to transitory income fluctuations. This implies that the volatility of optimal expected inflation decreases with income persistence.

Next, we consider an extension of the benchmark model to a production economy, in which physical capital is combined with the labor supply of young individuals to produce consumption goods. In this richer model, money is held as a store-of-value only if it provides the same expected return as capital. As a result, monetary policy is more restricted, but still can improve allocations via its effect on the value of nominal assets. Despite this richer structure, the same qualitative results are obtained for optimal monetary policy as in our simpler endowment economy.

Finally for the production economy, we show that the optimal monetary policy is well approximated by an inflation targeting (IT) rule that sets the expected future inflation at a target that is an increasing function of current inflation and current output. This kind of targeting policy, is often favoured by central banks due to the uncertainty surrounding economic mechanisms in the real economy, or uncertainty associated with data revisions. Another potential advantage of using targeting policies is the alleviation of the "inflation bias" that stems from the time-inconsistency problem faced by the monetary authority.[1]

An important contribution of policy rules is their stabilizing effect on future expectations and subsequently on long term decisions. Doepke and Schneider (2006) have shown that monetary policy can have sizable welfare consequences in an economy with heterogeneous sectors and nominal assets, via redistributive effects of inflation. Meh and Terajima (2008) have extended this insight beyond aggregate sectors and shown that different monetary

[1] Since the 1990s, 32 central banks announced inflation targeting as their monetary policy framework. See Walsh (1998) and Woodford (2003) for reviews of inflation targeting policy regimes. Ball and Sheridan (2003) provide a list of central banks that adopted inflation targeting, as well as timing details and performance evaluations for this policy change.

policy regimes can lead to various patterns of wealth redistribution between households of different age groups. These findings suggest that price-level uncertainty in a monetary policy regime can have a significant impact on expected returns of long-term nominal assets (such as mortgages[2]) and on ex-post wealth redistributions between generations. This is where policy rules are of key importance as they reduce price uncertainty and improve conditions regarding long run planning. Our model captures the key elements of the redistributive nature of monetary policy from a household perspective by incorporating nominal contracts, heterogenous households and aggregate risk.

The paper contributes to macroeconomic theory and monetary policy analysis along several dimensions. First, it shows, using a tractable model, the consumption smoothing behavior in an OLG environment with aggregate income shocks can lead to suboptimal variation in savings rates. This result contrasts with the "permanent income hypothesis" literature in which the absence of agent heterogeneity makes the consumption smoothing behavior fully efficient.[3] Furthermore, our paper enriches the insights of the "income fluctuations problem" which focuses on the average or steady-state inefficiency of savings behavior in models with uninsurable idiosyncratic income risk (but no aggregate risk).[4] The model in this paper focuses on the savings behavior under aggregate uncertainty, income heterogeneity and incomplete risk-sharing, providing a rich yet tractable framework for monetary policy analysis.

To our knowledge, there is very little research on monetary policy in a stochastic OLG environment. Perhaps surprisingly, most of the previous research on monetary policy in OLG models focused exclusively on deterministic models. Suboptimality of positive inflation was one of the main findings of that literature.[5] Akyol (2004) also finds positive optimal inflation in an environment with infinitely lived agents, who are subject to uninsurable idiosyncratic endowment risk and borrowing constraints. With no aggregate uncertainty, the price level in Akyol's model increases over time in a deterministic fashion. In our model, we provide a full characterization of optimal monetary policy under aggregate uncertainty. A recent paper by

[2] In the US, Mortgage debt of households is quite sizable reaching one GDP (Source: Economic Report of the President, (2010)).

[3] The fundamental idea was proposed by Milton Friedman, see Friedman (1957).

[4] Aiyagari (1994) shows that with uninsurable idiosycratic income risk (but no aggregate risk), households facing a constant rate of return on their savings, tend to oversave for precautionary reasons. See also Sargent and Linquist (2004), chapter 17 and references therein. Krusell and Smith (1998) add aggregate uncertainty, however, they do not focus on optimal policy.

[5] See, for example, Wallace (1992) or Champ and Freeman (2001).

Bhattacharya and Singh (2010) is related to ours. The authors use an overlapping generations endowment economy model in which spacial separation and random reallocation create an endogenous demand for money. Bhattacharya and Singh focus attention on comparing welfare implications of two different monetary policy rules, under various assumptions regarding the persistence of shocks: one with a constant growth rate of money, and another with a constant inflation rate. Our focus is different: we characterize the optimal monetary policy, which implies time-varying inflation and money growth rates. In our model with productive capital we find that inflation targeting rules closely approximate the optimal monetary policy, a result related to their conclusion. Overall, the findings of Bhattacharya and Singh (2010) complement our results.

The paper proceeds as follows. Section 2 introduces and analyzes the endowment economy with fiat money as the only asset. In Section 3, the model is extended to a production economy with capital. Section 4 contains concluding remarks. Proofs and derivations are collected in the appendices.

2. An OLG Model With Fiat Money

In this section, we study a two-period overlapping-generations endowment economy in which fiat money is the only asset. This simple environment allows an analytical characterization of the optimal monetary policy. In the model, the young individuals use money to save for the time when they are old. Monetary policy affects real returns on savings via its effect on expected inflation. Given asset market incompleteness, monetary policy has the potential to improve the average welfare in the economy.[6]

A. The Environment

There is a unit measure of identical individuals born in every period. Each generation lives for two periods. A young person born in period t is endowed with w_t units of a perishable consumption good in period t and zero units in period $t+1$. The endowment w_t is random and represents the only source of uncertainty in the model. The log of the endowment follows

[6] Markets are incomplete for two reasons. First, the overlapping-generations structure implies that newborn individuals cannot insure against the endowment risk. Second, young individuals, who save in the form of a noncontingent asset, cannot fully insure against rate-of-return risk.

a first-order autoregressive process:

$$\ln w_t = \rho \ln w_{t-1} + \varepsilon_t \ ,$$

where ε_t are i.i.d. draws from a zero-mean normal distribution with standard deviation σ.

The single asset in the economy is fiat money supplied by the government. In period 1 there is an initial old generation that has no endowment and holds M_0 units of the money stock.

The timing of events is as follows. At the beginning of period t the old generation holds M_{t-1} units of fiat money acquired in the previous period. Before the current endowment w_t is realized, the government prints (or destroys) money in the amount of $M_t - M_{t-1}$, and distributes it evenly among the old individuals via lump-sum transfer (or tax, if negative) $T_t = M_t - M_{t-1}$.[7] The assumption that monetary transfers occur before the realization of the current endowment, reflects the limited ability of the government's policy to react to current shocks in the economy, and implies an incomplete degree of control over the price level. After the realization of the current endowment, w_t, the young agents consume c_t^y units of their endowment. The remaining goods, $w_t - c_t^y$, are exchanged for M_t^d units of money at price P_t. Thus, a young person born in period t, solves the following problem:

$$(1) \qquad \max_{c_t^y, c_{t+1}^o, M_t^d} \quad u(c_t^y) + \beta E_t u(c_{t+1}^o)$$

subject to

$$\begin{aligned} (2) \qquad P_t c_t^y + M_t^d &\leq P_t w_t \ , \\ (3) \qquad P_{t+1} c_{t+1}^o &\leq M_t^d + T_{t+1} \ , \end{aligned}$$

where c_{t+1}^o is the person's consumption when old, T_t is the monetary transfer from the government in period t, β is the discount factor, and the period utility function $u(\cdot)$ satisfies the

[7] Appendix A2 shows that our results do not depend on the assumption that only the old receive the nominal transfer.

Inada conditions.[8] The operator E_t denotes the expected value conditional on the history of endowment realizations through the end of period t. Throughout the paper we use the following functional form for the period utility function: $u(c) = c^{1-\gamma}/(1-\gamma)$, where $\gamma > 0$ is the coefficient of risk aversion.

B. Monetary Equilibrium

Let μ_t denote the growth of money supply in the economy in period t, $\mu_t = \frac{M_t}{M_{t-1}}$. Monetary policy is defined as an infinite sequence of money growth rates, $\{\mu_t\}_{t=1}^{\infty}$ as functions of corresponding state histories.

Definition 1. *Given a monetary policy $\{\mu_t\}_{t=1}^{\infty}$ and initial endowment of money, a monetary equilibrium for this economy is a set of prices $\{P_t\}_{t=1}^{\infty}$ and allocations $\left\{c_t^y, c_t^o, M_t^d\right\}_{t=1}^{\infty}$, such that for all $t = 1, 2, 3, ...$*

1. *allocations c_t^y, c_{t+1}^o and M_t^d solve the generation t's problem (1)-(3), and*
2. *the good and money markets clear:*

$$\begin{aligned} c_t^y + c_t^o &= w_t \ , \\ M_t^d &= M_t \ . \end{aligned}$$

In the next subsection we characterize the optimal allocation and derive the optimal monetary policy that implements it as a monetary equilibrium.

C. Optimal Monetary Policy

To find the optimal monetary policy, we first define the social welfare function and solve the social planner's problem for the optimal allocation. We then ask whether this allocation can be implemented as a monetary equilibrium.

[8] Throughout the paper all variables are random functions of histories of endowment realizations. To keep notation simple the explicit state-history notation is omitted.

The Social Planner's Problem

The social planner is assumed to treat all generations equally. Let the average (ex-post) utility over T periods be:

$$
\begin{aligned}
V_T &= \frac{1}{T}\left[\beta u(c_1^o) + \left[\sum_{t=1}^{T-1}\left[u(c_t^y) + \beta u(c_{t+1}^o)\right]\right] + u(c_T^y)\right] \\
&= \frac{1}{T}\sum_{t=1}^{T}\left[u(c_t^y) + \beta u(c_t^o)\right] \ . \qquad (4)
\end{aligned}
$$

We define the social welfare function as

$$
\lim_{T\to\infty} \inf E\left[V_T\right] \ . \qquad (5)
$$

This welfare criterion treats all generations equal by attaching the same welfare weight to the expected utility of each generation.

The social planner maximizes (5) subject to the resource constraint for all periods:

$$
c_t^y + c_t^o \leq w_t, \text{ for } t = 1, 2, ... \ .
$$

We show in Appendix A1 that the solution to this problem is the sequence of consumptions $\{c_t^y, c_t^o\}_{t=1}^{T}$ such that in each period the marginal utilities of consumption of the young and of the old are equal:

$$
\begin{aligned}
u'\left(c_t^y\right) &= \beta u'(c_t^o) \ , \\
c_t^y + c_t^o &= w_t \ .
\end{aligned}
$$

The dynamic behavior of individual savings decisions and of the optimal allocation of consumption across generations is the determinant of the properties of optimal monetary policy in the model. For the case of a constant relative risk aversion (CRRA) period utility function,

$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, the first-best allocation is:

$$(6) \quad c_t^y = \frac{1}{1+\beta^{\frac{1}{\gamma}}} w_t \ ,$$

$$(7) \quad c_t^o = \frac{\beta^{\frac{1}{\gamma}}}{1+\beta^{\frac{1}{\gamma}}} w_t \ ,$$

for all $t = 1, 2, \ldots$. Note that the first-best allocation calls for a constant savings rate of the young to be equal to $\frac{w_t - c_t^y}{w_t} = \frac{\beta^{\frac{1}{\gamma}}}{1+\beta^{\frac{1}{\gamma}}}$.

There are two reasons for using the undiscounted welfare function (4) rather than the discounted one,

$$(8) \quad V_T = u(c_1^o) + \sum_{t=1}^{T} \beta^{t-1} \left[u(c_t^y) + \beta u(c_{t+1}^o) \right] .$$

The discounted social welfare function (8) implies that the optimal consumption is equally divided between the young and the old: $c_t^y = c_t^o = \frac{1}{2} w_t$. This pattern of lifetime consumption does not maximize ex-ante utility of young individuals in an OLG economy, because they discount old-age consumption and would prefer higher expected consumption when young. In contrast, the consumption allocation (6) and (7), implied by the undiscounted welfare function, maximizes the unconditional expected utility of any given generation (except the initial old). Hence, it is the unique *ex-ante* optimal allocation. Secondly, without endowment uncertainty, the allocation (6) and (7) is implementable as a market equilibrium with a constant money stock. Hence, by using the undiscounted welfare function we circumvent the issue of "dynamic inefficiency" common in non-stochastic OLG models and focus on the dynamic properties of optimal monetary policy arising in response to shocks.[9]

Implementing the Optimal Allocation as a Monetary Equilibrium

Suppose the first-best allocation can be implemented in a monetary equilibrium. Any monetary equilibrium must satisfy the following two necessary and sufficient first-order con-

[9] In Diamond (1965) the "dynamic inefficiency" stems from population growth affecting the discount factor in the social welfare function in an OLG-type setup.

ditions[10]:

$$
\begin{aligned}
(9) \quad u'\left(w_t - \frac{M_t^d}{P_t}\right) &= \beta E_t\left[u'\left(\frac{M_t^d + T_{t+1}}{P_{t+1}}\right)\frac{P_t}{P_{t+1}}\right], \\
(10) \quad \frac{P_t}{P_{t+1}} &= \frac{M_t}{M_{t+1}}\frac{w_{t+1} - c_{t+1}^y}{w_t - c_t^y}.
\end{aligned}
$$

Equation (9) is a standard intertemporal Euler condition. Equation (10) is derived from the budget constraint of the young (2) holding with equality, and from the money market clearing condition $M_t^d = M_t$.

In equilibrium monetary policy can affect savings by varying the expected real rate of return on money. The ex-post real rate of return on money is given by the inverse of the inflation rate, $\frac{P_t}{P_{t+1}}$ and is affected by the growth rate of money $\mu_{t+1} = \frac{M_{t+1}}{M_t}$. Due to our timing assumption on monetary injections, the growth rate of money μ_{t+1} is determined before w_{t+1} and P_{t+1} are realized. Thus, equation (10) implies

$$\frac{M_{t+1}}{M_t} = E_t\left[\frac{P_{t+1}}{P_t}\frac{w_{t+1} - c_{t+1}^y}{w_t - c_t^y}\right]$$

where the ratio $\frac{w_{t+1} - c_{t+1}^y}{w_t - c_t^y}$ is the growth rate of savings in this economy between periods t and $t+1$. Changes in the growth rate of money, μ_{t+1}, affect both the ex-ante return on money, and the expected growth rate of savings.

Before we characterize the optimal monetary policy in this economy, it is instructive to look at a related OLG economy in which the rate of return on savings is technological and cannot be changed. Suppose that instead of money, agents have access to a storage technology, which gives a fixed real return of R for every unit of goods invested. The first-order conditions of a young generation in this modified economy are:

$$
\begin{aligned}
(11) \quad u'\left(c_t^y\right) &= \beta E_t\left[u'\left(c_{t+1}^o\right)R\right], \\
(12) \quad c_t^y &= w_t - s_t \\
(13) \quad c_{t+1}^o &= s_t R.
\end{aligned}
$$

[10]Sufficiency follows from the concavity of the decision problem.

where s_t is the amount of real goods stored by the young person in period t. Notice that the old age consumption c_{t+1}^o is completely independent of the endowment realization in period $t+1$, w_{t+1}. This means that the young agents in this economy bear all the endowment risk, while the old agents face no uncertainty. This is clearly not an optimal allocation, since it precludes any risk sharing between generations. If the ex-post real rate of return on savings was an increasing function of the growth rate of endowment $\frac{w_{t+1}}{w_t}$, then the degree of risk sharing between generations would increase.

In our model this can be achieved by an appropriately set monetary policy. If the policy is such that ex-post inflation is decreasing in $\frac{w_{t+1}}{w_t}$, then the real rate of return on money is increasing in $\frac{w_{t+1}}{w_t}$. Notice that with a mean reverting stochastic process for endowment, this pattern of adjustment in ex-post inflation implies a procyclical expected inflation. When w_t is high, the expected value of the ratio $\frac{w_{t+1}}{w_t}$ is low, which means that the expected inflation must be high. On the contrary, when w_t is low, the expected value of the ratio $\frac{w_{t+1}}{w_t}$ is high, and the expected inflation is low. Thus, the optimal monetary policy in our economy must lead to a procyclical expected inflation rate. Our analysis of the optimal monetary policy below confirms this conclusion.

We combine equilibrium conditions (9)-(10) with the first-best allocation (6) and (7) to obtain the expression for optimal money growth

$$\text{(14)} \quad \frac{M_{t+1}}{M_t} = E_t\left[\left(\frac{w_{t+1}}{w_t}\right)^{1-\gamma}\right],$$

and prices

$$\text{(15)} \quad \frac{P_{t+1}}{P_t} = \frac{M_{t+1}}{M_t}\frac{w_t}{w_{t+1}} = E_t\left[\left(\frac{w_{t+1}}{w_t}\right)^{1-\gamma}\right]\frac{w_t}{w_{t+1}} .$$

Given the assumption of log normality of the endowment process, equations (14) and (15) imply that

$$(16) \quad m_{t+1} - m_t = \frac{(1-\gamma)^2\sigma^2}{2} + (1-\rho)(\gamma-1)\omega_t,$$

$$(17) \quad p_{t+1} - p_t = \frac{(1-\gamma)^2\sigma^2}{2} + \gamma(1-\rho)\omega_t - \varepsilon_{t+1} \ ,$$

where $m_t \equiv \ln M_t$, $p_t \equiv \ln P_t$, $\omega_t \equiv \ln w_t$.

Equations (16) and (17) together with initial conditions fully characterize the dynamics of money growth and the price level in the equilibrium that implements the optimal consumption allocation. It also follows from these equations that the monetary equilibrium and the optimal monetary policy are unique (up to initial conditions) for any sequence of endowments $\{\omega_t\}$. Equation (17) implies the following equation for expected inflation as a function of endowment:

$$(18) \quad E_t\left[p_{t+1} - p_t\right] = \frac{(1-\gamma)^2\sigma^2}{2} + \gamma(1-\rho)\omega_t.$$

While the cyclical properties of the optimal money growth rate depend on the risk aversion parameter γ, the expected inflation is procyclical for any positive γ. This follows from equations (16) and (18). For example, in the case of log utility, $\gamma = 1$, optimal money growth is zero but expected inflation is procyclical. Furthermore equation (15) implies that the ex-post inflation rate is a decreasing function of the growth rate of endowment $\frac{w_{t+1}}{w_t}$ and, controlling for the growth rate of money, is countercyclical. This is an implication of the timing assumption for monetary injections: money supply in period $t+1$ is independent of the endowment realization in that period. This means that high income realizations of ω_{t+1} will lower the price level p_{t+1} and the ex-post inflation rate.

We summarize the main properties of price level dynamics under the optimal monetary policy in the following proposition:

Proposition 1

1. *The average inflation under the optimal policy is positive,* $\bar{\pi} = \frac{(1-\gamma)^2\sigma^2}{2}$*, and increasing with the size of uncertainty, as long as* $\gamma \neq 1$*.*

2. *Expected inflation is positively correlated with the current endowment.*
3. *The variance of expected inflation is decreasing in the persistence of the endowment process, ρ. If the endowment follows a random walk, $\rho = 1$, then the optimal expected inflation is constant: $E_t [p_{t+1} - p_t] = \bar{\pi}$.*

Proof

While the proof of this proposition follows immediately from the equation (18), Appendix A3 provides derivations of the equations (16) and (17).

Another way to understand the rationale for the properties of the optimal policy is by looking at their effects on savings rates. Recall from equations (6) and (7) that the first-best allocation corresponds to the constant savings rate, $\frac{\beta^{\frac{1}{\gamma}}}{1+\beta^{\frac{1}{\gamma}}}$. In a monetary equilibrium, the savings rate depends on the expected return to money $E_t [p_t - p_{t+1}]$, which is the negative of the expected inflation. The monetary authority sets expected inflation, by appropriately choosing the rate of money growth, to stabilize the equilibrium savings rate at the optimal level. The three properties of the optimal monetary policy describe how expected inflation must be set to achieve the first-best allocation.

Property 1 is due to asset market incompleteness, implying that individuals cannot perfectly insure themselves against endowment risk. In the face of uncertainty about future income, risk-averse individuals have an incentive to self-insure by smoothing consumption across time. Without positive trend inflation, they tend to save on average more than optimal for precautionary reasons, as in Aiyagari (1994).[11] The positive average inflation serves as a tax on savings, which discourages oversaving. The log-utility case ($\gamma = 1$) is a notable exception from this rule: the optimal long-run inflation rate is precisely zero for this special case. As is well known, with log-utility individuals do not vary their savings rates in response to uncertainty about future rates of return. As a result, the optimal inflation rate is zero.[12]

[11] More precisely, Property 1 says that inflation under optimal policy with uncertainty is higher than without uncertainty (zero in this case).

[12] It might seem strange at first, that the optimal inflation rate is positive for both γ greater and less than one. It is well known that with a CRRA utility function the sign of the relationship between (uncompensated) changes in the rate of return on savings and the supply of savings depends crucially on the value of the risk aversion parameter γ. This is because the relative strength of two opposing effects induced by changes in the rate of return: the income effect and the substitution effect, depend on γ. In our case however, changes in the long-run inflation rate do not have income effects, because the newly created money is rebated to the households. As a result, the higher inflation rate reduces savings for both $\gamma > 1$ and $\gamma \in (0, 1)$.

According to Property 2, a positive correlation between the expected inflation and income implies a high (low) tax on savings when income is high (low). This discourages individuals from varying savings rates to smooth consumption over time and thereby stabilizes the savings rate.[13] Hence Property 2 of the optimal monetary policy rectifies the cyclical component of socially suboptimal precautionary savings, whereas Property 1 dampens its average component.

Finally, Property 3 implies that with a higher endowment persistence there is a smaller difference between the endowment of current-period young relative to next-period young and thus lower incentives to vary the savings rate. Furthermore, we see less variation in the marginal utility of consumption between the young and the old. As a result, the optimal expected inflation has to vary less to discourage consumption smoothing. In the limit, when income follows a random walk, the optimal expected inflation is constant.

It is relatively straightforward to show that the OLG structure is not essential for our results. The mechanism through which monetary policy implements reallocation of resources under aggregate uncertainty requires the existence of non-contingent nominal assets and ex-post income heterogeneity across agents and time. In the extended version of the paper, Kryvtsov, Shukayev, and Ueberfeldt (2007), we study an infinite-horizon model, in which these elements make nominal non-contingent assets essential and create demand for trades that facilitate risk sharing. The results in the infinite-horizon model are equivalent to those in the OLG model of this section.

In the next Section we extend our simple model to a production economy with capital.

3. An OLG Economy With Capital and Money

To keep things analytically tractable, the model in the previous section has only one asset, fiat money, and an exogenous income process. In this section, we present a richer model, in which households can save by accumulating capital in addition to money, and physical capital can be combined with the labor endowment of the young to produce consumption goods. Since capital can be used as a store of value, money has to promise the

[13] The response of the savings rate to expected real return depends on the relative risk aversion parameter γ. When $\gamma > 1$, the savings rate is decreasing in expected return, while with $\gamma \in (0, 1)$ the savings rate is increasing in expected return. However, we find for either case that the expected inflation must be increasing in income in order to *stabilize* the savings rate. A constant money stock creates too little procyclicality of expected inflation when $\gamma > 1$, and too much when $\gamma \in (0, 1)$. As a result the optimal growth rate of money is procyclical when $\gamma > 1$ and countercyclical when $\gamma \in (0, 1)$ as is clear from equation (16).

same expected return, to be held along with capital. We derive the optimal monetary policy which implements the first-best allocation, using the same welfare criterion as before. We find that the qualitative results and the intuition for the endowment economy with money carry over to the production economy with money and capital.

A. The Environment

There is a unit measure of agents born every period. All individuals of the same generation are identical in all respects. Every generation lives for two periods. At the beginning of period t, the young generation is endowed with a unit of time ($N = 1$) that can be used for work. The old own the entire stock of capital K_{t-1} plus the entire stock of money M_{t-1}. The government prints (destroys) new money in the amount $M_t - M_{t-1}$, and allocates it equally among the old with a lump-sum transfer (or tax) $T_t = M_t - M_{t-1}$.

After the money transfer T_t takes place, a productivity shock A_t is realized. The young inelastically supply their working time ($N_t = N$) and rent capital from the old to produce output Y_t using a Cobb-Douglas production technology: $Y_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha} = A_t K_{t-1}^{\alpha}$. They use part of that output to pay rental income $r_t K_{t-1}$ to the old. The remainder of their income plus a government lump-sum real subsidy G_t, is used for consumption c_t^y and for investment into capital K_t and money M_t^d.

To ensure the existence of a monetary equilibrium in this model, the government must redistribute income from the old to the young in every period. In the absence of redistribution, standard values of the capital income share α imply that the young generation's share of income is too small for them to invest into both capital and nominal assets.[14] Since our focus is on the dynamic properties of monetary policy, we delegate the role of steady-state income redistribution to the fiscal policy. For this purpose, we assume that the government subsidy G_t is paid only to the young. The subsidy is financed by taxing consumption of young and old at a fixed rate τ, i.e., $G_t = \tau\left(c_t^y + c_t^o\right)$.[15]

[14] Alternatively, sufficiently small values of α make income redistribution unneccessary.

[15] A fixed lump-sum tax alternative has an unattractive feature of being completely independent of income, which might lead to the transfer being infeasible when income realizations are particularly low. Further, we chose a consumption tax rather than an income tax because income taxes distort intertemporal investment decisions, which makes it impossible for the monetary policy to attain the first best. Since our focus in this paper is on the optimal monetary policy, a fixed consumption tax is a more convenient redistributive tool for our purposes.

Thus, the budget constraint of the young in period t is

$$(19)\quad (1+\tau)\, c_t^y + \frac{M_t^d}{P_t} + K_t \leq A_t K_{t-1}^{\alpha} - r_t K_{t-1} + G_t.$$

The current old, on the other hand, in period t consume everything they have:

$$(20)\quad (1+\tau)\, c_t^o = r_t K_{t-1} + \frac{M_{t-1}^d + T_t}{P_t}.$$

We assume that the log of the productivity shock, $a_t = \ln A_t$, follows a first-order autoregressive process:

$$a_t = \rho a_{t-1} + \varepsilon_t \ ,$$

where ε_t are i.i.d. draws from normal distribution $N(0, \sigma^2)$. The productivity shock process is the only source of uncertainty in the model.

The problem of the young in period t is the following:

$$\max \quad u(c_t^y) + \beta E_t u(c_{t+1}^o)$$

subject to (19) and the period-$(t+1)$ version of (20).

DEFINITION 2. *Given a monetary policy* $\{\mu_t\}_{t=1}^{\infty}$ *and initial endowments of capital and money, a monetary equilibrium in the production economy is a set of prices* $\{P_t\}_{t=1}^{\infty}$ *and allocations* $\left\{c_t^y, c_t^o, K_t, M_t^d\right\}_{t=1}^{\infty}$, *such that for all* $t = 1, 2, 3, \ldots$

1. *allocations* c_t^y, c_{t+1}^o, K_t *and* M_t^d *solve the generation* t*'s problem, and*
2. *the good, labor, capital and money markets clear. In particular, good market clearing condition is*

$$c_t^y + c_t^o + K_t = A_t K_{t-1}^{\alpha}.$$

Similarly to the endowment economy, a relatively simple equilibrium system can be derived for the CRRA utility function $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$. In this case, equilibrium sequences of real money balances $x_t = \frac{M_t}{P_t}$ and capital K_t satisfy:

(21)
$$\left(\frac{(1-\alpha+\tau)\,A_t K_{t-1}^{\alpha} - x_t - (1+\tau)\,K_t}{1+\tau}\right)^{-\gamma} = \beta E_t\left[\left(\frac{\alpha A_{t+1}K_t^{\alpha} + x_{t+1}}{1+\tau}\right)^{-\gamma} \alpha A_{t+1}K_t^{\alpha-1}\right],$$

(22)
$$\left(\frac{(1-\alpha+\tau)\,A_t K_{t-1}^{\alpha} - x_t - (1+\tau)\,K_t}{1+\tau}\right)^{-\gamma} = \beta E_t\left[\left(\frac{\alpha A_{t+1}K_t^{\alpha} + x_{t+1}}{1+\tau}\right)^{-\gamma} \frac{x_{t+1}}{x_t\mu_{t+1}}\right].$$

To complete the description of the model it remains to specify a monetary policy. Again, we focus on the optimal monetary policy.

B. Optimal Monetary Policy

We maintain the same social welfare criterion as in Section 2. For given initial endowments of capital, the social planner solves the following problem:

$$\max \lim_{T\to\infty} \inf E\left[\frac{1}{T}\sum_{t=1}^{T}\left[u(c_t^y) + \beta u(c_t^o)\right]\right]$$

subject to the resource constraint:

$$c_t^y + c_t^o + K_t \le A_t K_{t-1}^{\alpha}, \quad \forall t\ .$$

The first-order conditions for this problem are:

$$\begin{aligned} u'\left(c_t^y\right) &= \lambda_t\ , \\ \beta u'\left(c_t^o\right) &= \lambda_t\ , \\ \lambda_t &= \alpha E_t \lambda_{t+1} A_{t+1} K_t^{\alpha-1}, \end{aligned}$$

where λ_t is the Lagrange multiplier on the resource constraint.

With CRRA utility function the optimal allocation of consumption is

$$(23)\quad c_t^y = \frac{1}{1+\beta^{\frac{1}{\gamma}}}\left(A_t K_{t-1}^{\alpha} - K_t\right),$$

$$(24)\quad c_t^o = \beta^{\frac{1}{\gamma}} c_t^y = \frac{\beta^{\frac{1}{\gamma}}}{1+\beta^{\frac{1}{\gamma}}}\left(A_t K_{t-1}^{\alpha} - K_t\right),$$

where the optimal capital sequence satisfies

$$(25)\quad \left[A_t K_{t-1}^{\alpha} - K_t\right]^{-\gamma} = E_t\left\{\left[A_{t+1} K_t^{\alpha} - K_{t+1}\right]^{-\gamma} \alpha A_{t+1} K_t^{\alpha-1}\right\}.$$

To find the policy that implements the optimal allocation, we plug the consumption allocations (23) and (24) into the budget constraints (19) and (20) holding with equality, and into the monetary equilibrium conditions (21) and (22) to obtain:[16]

$$(26)\quad \left[A_t K_{t-1}^{\alpha} - K_t\right]^{-\gamma} = E_t\left\{\left[A_{t+1} K_t^{\alpha} - K_{t+1}\right]^{-\gamma} \alpha A_{t+1} K_t^{\alpha-1}\right\},$$

$$(27)\quad \left[A_t K_{t-1}^{\alpha} - K_t\right]^{-\gamma} = \frac{1}{\mu_{t+1}} E_t\left[\left[A_{t+1} K_t^{\alpha} - K_{t+1}\right]^{-\gamma} \frac{x_{t+1}}{x_t}\right].$$

where real money balances satisfy[17]

$$(28)\quad x_t = (1+\tau)\frac{\beta^{\frac{1}{\gamma}}}{1+\beta^{\frac{1}{\gamma}}}\left(A_t K_{t-1}^{\alpha} - K_t\right) - \alpha A_t K_{t-1}^{\alpha}.$$

Equation (26) is identical to the Euler equation for the optimal allocation (25), which means that the optimal monetary policy ensures optimal capital investment. Equation (27) implies that in equilibrium the ex-ante return on nominal assets equals the ex-ante return on capital. The sequence of money growth rates that implements the first-best allocation is given by

[16] Recall that M_{t+1} is determined before the $t+1$ productivity shock is realized, so $E_t\left[1/\mu_{t+1}\right] = 1/\mu_{t+1}$.

[17] In general $x_t = \frac{\beta^{\frac{1}{\gamma}}(1+\tau)}{1+\beta^{\frac{1}{\gamma}}}\left(A_t K_{t-1}^{\alpha} - K_t\right) - \alpha A_t K_{t-1}^{\alpha}$ does not always have to be positive. We however sidestep this issue by setting $(1+\tau)$ high enough so that x_t is always positive in our simulations. More specifically, we set it at $(1+\tau) = \frac{\beta^{\frac{1}{\gamma}}}{1+\beta^{\frac{1}{\gamma}}}$ so that $x_t = (1-\alpha) A_t K_{t-1}^{*\alpha} - K_t^*$. As long as x_t is positive, the value of τ does not affect the dynamic aspects of the model, which is the focus of this study.

(27). The inflation rate under the optimal policy is then

$$(29)\quad \pi_{t+1} = \ln \frac{P_{t+1}}{P_t} = \ln \mu_{t+1} + \ln x_t - \ln x_{t+1}$$

and the expected optimal inflation

$$(30)\quad E_t\left[\pi_{t+1}\right] = \ln \mu_{t+1} + \ln x_t - E_t \ln x_{t+1}.$$

C. Properties of the Optimal Monetary Policy

In this section we characterize the optimal monetary policy and check whether it has the same dynamic properties as in the simple endowment economy of Section 2. Unlike the simple economy case however, the economy with capital cannot be solved fully analytically. As a result, we use a combination of first-order approximation techniques and numerical computations to deduce the dynamic properties of optimal inflation. We start by stating the following proposition:

Proposition 2

1. *The average inflation under the optimal policy is positive, as long as* $\gamma \neq 1$.
2. *Up to the first-order of approximation, the optimal expected inflation is determined according to the following equation:*

$$(31)\quad E_t \pi_{t+1} = \eta_\pi \pi_t + \eta_y y_t + \eta_a a_{t-1},$$

where $y_t = \ln\left(\frac{Y_t}{\bar{Y}}\right) = \ln\left(\frac{A_t K_{t-1}^\alpha}{\bar{Y}}\right)$ *is the log-deviation of output from its steady state value,* $a_t = \ln A_t$ *is the log-productivity term and the elasticity coefficients* $\eta_\pi \geq 0$, $\eta_y \geq 0$ *and* η_a *are given in Appendix A4.*

3. *In the special case of the logarithmic utility function* $(\gamma = 1)$, *the exact analytical solution for expected inflation is:*

$$(32)\quad E_t \pi_{t+1} = \alpha\rho\pi_t + (1-\alpha)(1-\rho)\, y_t,$$

which implies that:

(a) *Expected inflation is positively correlated with the current output*

(b) *The responsiveness of expected inflation to current output is decreasing in the persistence of the productivity process, ρ. If productivity follows a random walk, $\rho = 1$, then the optimal expected inflation is independent of output.*

(c) *As long as $\alpha > 0$ and $\rho > 0$, the optimal expected inflation is positively correlated to current inflation, with a response elasticity which is increasing in the persistence of the productivity process, ρ.*

Proof

See Appendix A4.

Equation (31) represents a policy rule that determines the target of monetary policy in this model, i.e. expected inflation, as a function of current inflation, output and the lagged productivity level.

An interesting property of the law of motion governing expected inflation is that it is similar to a standard inflation targeting policy rule up to the lagged productivity term $\eta_a a_{t-1}$. We will show that for a big set of parameter values, this last term is relatively unimportant for the dynamics of expected inflation. Thus, the simple rule

$$(33) \quad E_t \pi_{t+1} = \eta_\pi \pi_t + \eta_y y_t,$$

provides a good approximation of optimal monetary policy. This property is useful from a practical point of view, since many central banks today have adopted an inflation-targeting policy that is based on setting the target for future inflation as a function of current inflation and output.[18]

The exact solution for the dynamics of the optimal expected inflation obtained for the case of a logarithmic utility function is quite remarkable. In this case, the optimal expected

[18] If fluctuations in real interest rate are second order, equation (33) can be written in the form of a Taylor rule:

$$i_t = \bar{\imath} + \eta_\pi \pi_t + \eta_y y_t.$$

inflation is a weighted average of current inflation and current output

(34) $E_t\pi_{t+1} = \alpha\rho\pi_t + (1-\alpha)(1-\rho)\, y_t.$

Note that the weight on inflation is increasing in ρ from zero to α, while the weight on output is decreasing in ρ from $(1-\alpha)$ to zero, in particular, if $\alpha = 0$ the solution coincides to the one for the endowment economy. Thus, the dynamic properties 2 and 3 of optimal monetary policy in the endowment economy transfer perfectly to this case of the production economy. Namely, expected inflation is procyclical and its responsiveness to output fluctuations is decreasing in output persistence.

The dependence of expected future inflation on current inflation is a new aspect, which was not present in the simple endowment economy. In the endowment economy monetary policy was targeting the real interest rate, which is a function of total output. In the production economy monetary policy is targeting the real return on nominal assets relative to the real return on capital. The elasticity of next period's return on capital with respect to current productivity is $\alpha\rho$. Hence, given the change in output due to a change in productivity, expected inflation needs to increase by an additional amount of $\alpha\rho\pi_t$ to bring the real interest rate at par with the increased ex-ante real rate of return on capital. If the ex-ante return on capital is constant (i.e. the capital income share in production, α, is zero) or productivity fluctuations are i.i.d. ($\rho = 0$), then the current inflation term in equation (32) disappears.

For other values of γ we use numeric simulations to demonstrate that the dynamic properties 2 and 3 of optimal monetary policy in the endowment economy hold also in the production economy. Specifically, we assign structural parameter values as summarized in Table 1 and plot the elasticity parameters η_π, η_y and η_a in equation (31) against different values of persistence of productivity shocks, ρ.

Figure 1 plots the elasticities of expected inflation with respect to current inflation and output, $\eta_\pi$and η_y respectively. It shows that for $\gamma = 0.5$, $\gamma = 1.5$ and $\gamma = 4$ the general pattern is the same as for the logarithmic case ($\gamma = 1$): the coefficient on inflation, η_π, is positive and increasing in persistence, ρ, while the coefficient on output, η_y, is positive and decreasing in ρ, though generally not to zero.

Figure 2 plots the elasticity of expected inflation with respect to the lagged productivity shock, η_a, as a function of ρ. From the figure we can see that for γ close to 1, the coefficient η_a is generally small, but not necessarily zero. For higher $\gamma = 4$ the elasticity coefficient η_a becomes larger in absolute value, thus increasing the importance of the lagged productivity term for optimal inflation dynamics.

Hence, the main properties of optimal monetary policy that we documented for a simple endowment economy carry over to a more general economy with production and other assets. In the remainder of this section we check the accuracy of our linear approximation results by solving the non-linear version of the model.

D. Non-linear simulations of optimal monetary policy

We set the standard deviation of productivity innovations at $\sigma = 0.16$,[19] and use a collocation method with a dense grid to solve equation (26) for the optimal capital sequence. Once we know the optimal capital sequence, we can use equations (23), (24), (28), (29) and (30) to solve for all the other variables, including expected inflation. Then we simulate a long ($T = 10,000$ periods) series of the optimal expected inflation and find residuals that are not explained by the linear model (31):

$$\xi_t = E_t\pi_{t+1} - \eta_\pi \pi_t - \eta_y y_t - \eta_a a_{t-1}.$$

We use the residuals to compute two summary statistics. First, we compute the fraction of the total sample variation in expected inflation accounted for by the sample variation in current inflation, output and lagged productivity:

$$\Omega_1 = 1 - \frac{\sum_{t=1}^{T}\left(\xi_t - \bar{\xi}\right)}{\sum_{t=1}^{T}\left(E_t\pi_{t+1} - \bar{\pi}\right)}.$$

This measure quantifies the accuracy of our first-order approximations results, which ignores higher order variations in expected inflation. The closer Ω_1 is to unity, the smaller is the approximation error of our analytic solution for expected inflation.

[19] Our estimates of σ from long-term U.S. and U.K. GDP data, range from 0.02 to 0.16 depending on how we detrend the data and the assumed stationarity of the income process. We chose a higher value from this range to get an upper bound on the importance of productivity fluctuations for the optimal expected inflation.

The second summary statistic is the fraction of total variation in expected inflation due to sample variation in current inflation and output only:

$$\Omega_2 = 1 - \frac{\sum_{t=1}^{T} \left(\xi_t + \eta_a a_{t-1} - \bar{\xi}\right)}{\sum_{t=1}^{T} \left(E_t \pi_{t+1} - \bar{\pi}\right)}.$$

This shows how closely a simple linear inflation targeting rule (33) emulates the optimal expected inflation. The closer Ω_2 is to unity, the less important is the lagged productivity term in equation (31) for the dynamic behavior of expected inflation.

Tables 2 and 3 show Ω_1 and Ω_2 statistics for various values of ρ, and γ. As we can see from the results for Ω_1, the first-order approximated solution accounts for more than 99 percent of total variation in the expected inflation, except when the relative risk aversion (γ) is very far from unity and at the same time the productivity process is highly president ($\rho \to 1$). Similarly, the lagged productivity term accounts for only a small fraction of variation in the optimal expected inflation, as summarized by Ω_2.

We recap the following general results regarding the optimal monetary policy in the model with capital and money :

1. Average inflation under the optimal policy is positive.
2. Expected inflation is positively correlated with current income.
3. The degree to which expected inflation responds to income fluctuations is decreasing in the persistence of income fluctuations.
4. The dynamics of optimal inflation very closely resemble a simple inflation targeting rule in that expected future inflation is an increasing function of current inflation and output.

Overall, the results in this section confirm and enrich the insights obtained from the simple endowment economy in Section 2.

4. Conclusions

We explore the role of monetary policy in the environment with aggregate risk, incomplete markets and long-term nominal bonds. In a two-period overlapping-generations model with aggregate uncertainty and nominal bonds, optimal monetary policy attains the ex-ante Pareto optimal allocation. This policy implies a positive average inflation, a positive correlation between expected inflation and income, and an inverse relationship between the volatility of expected inflation and the persistence of income. The results extend to a more general environment with productive capital. The model with capital predicts that the dynamics of the optimal inflation resemble a simple inflation targeting rule very closely, which sets the target for future inflation as an increasing function of current inflation and output.

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Appendix

1. The Solution To the Social Planner's Problem

Suppose, for a given history of endowment realizations, $w^T = \{w_1, w_2, ..., w_T\}$, we are solving the following problem:

$$\begin{aligned} & \max V_T \\ \text{subject to} & : \\ (1) \qquad c_t^y + c_t^o & \leq w_t, \text{ for all } t = 1, 2, ..., T. \end{aligned}$$

The solution of this problem is $\{c_t^y, c_t^o\}_{t=1}^T$ such that:

$$\begin{aligned} u'(c_t^y) &= \beta u'(c_t^o), \\ c_t^y + c_t^o &= w_t. \end{aligned}$$

It is a pair of consumption functions $c_t^y = c^{y*}(w_t)$, and $c_t^o = c^{o*}(w_t)$. Given w_t, they are independent of T and of the realized endowment history w^T.

Let

$$V_T^* = \frac{1}{T} \sum_{t=1}^{T} \left[u(c^{y*}(w_t)) + \beta u(c^{o*}(w_t)) \right].$$

Let $\{c_t^y, c_t^o\}_{t=1}^T$ be any other sequence of consumptions that satisfies (1) in each period t, and let V_T be the corresponding average ex-post utility as defined in (4). Then $V_T^* \geq V_T$, since for all $t = 1, 2..., T$ we have

$$(2) \quad u(c^{y*}(w_t)) + \beta u(c^{o*}(w_t)) \geq u(c_t^y) + \beta u(c_t^o).$$

Taking expectation of $V_T^* - V_T$ with respect to realizations of w^T we have:

$$E[V_T^*] - E[V_T] \geq 0.$$

Taking the *liminf* with respect to T, we have

$$\lim_{T\to\infty} \inf \left(E\left[V_T^*\right] - E\left[V_T\right]\right) \geq 0.$$

Since the sequence $\{c_t^y, c_t^o\}_{t=1}^T$ was arbitrary, the stationary policy $c^{y*}(w), c^{o*}(w)$ attains the maximum of the expected average utility, $E\left[V_T\right]$, for all T.

2. Relaxing the assumption on monetary injections being given to the old

In the simple endowment economy of Section 2 we assumed that the entire monetary injection $M_t - M_{t-1}$ is given to the old agents only. Here we will generalize that assumption by assuming that the old agents get a fraction $\phi \in (0,1]$ of the monetary injection, while the young receive the rest. Thus the young person born in period t solves:

$$\max u(c_t^y) + \beta E_t u(c_{t+1}^o)$$

subject to

$$\begin{aligned} P_t c_t^y + M_t^d &\leq P_t w_t + T_t^y \\ P_{t+1} c_{t+1}^o &\leq M_t^d + T_{t+1}^o \ , \end{aligned}$$

where

$$\begin{aligned} T_t^y &= (1-\phi)\left(M_t - M_{t-1}\right) \\ T_{t+1}^o &= \phi\left(M_{t+1} - M_t\right). \end{aligned}$$

The first-order condition for this problem

$$u'\left(w_t + \frac{T_t^y}{P_t} - \frac{M_t^d}{P_t}\right) = \beta E_t\left[u'\left(\frac{M_t^d + T_{t+1}^o}{P_{t+1}}\right)\frac{P_t}{P_{t+1}}\right]$$

and the money market clearing condition is $M_t^d = M_t$, imply

$$u'\left(w_t - \frac{\phi M_t + (1-\phi) M_{t-1}}{P_t}\right) = \beta E_t\left[u'\left(\frac{\phi M_{t+1} + (1-\phi) M_t}{P_{t+1}}\right)\frac{P_t}{P_{t+1}}\right].$$

With a CRRA utility function, the first-best allocation (6), (7) implies the following expression for real return on money:

$$\begin{aligned}\frac{P_t}{P_{t+1}} &= \frac{\phi M_t + (1-\phi) M_{t-1}}{w_t - c_t^y}\frac{w_{t+1} - c_{t+1}^y}{\phi M_{t+1} + (1-\phi) M_t} \\ &= \frac{\phi M_t + (1-\phi) M_{t-1}}{\phi M_{t+1} + (1-\phi) M_t}\frac{w_{t+1}}{w_t}.\end{aligned}$$

It follows that the optimal monetary policy is implementable with the money stock growing according to the rule

$$(w_t)^{-\gamma} = E_t\left[(w_{t+1})^{-\gamma}\frac{\phi M_t + (1-\phi) M_{t-1}}{\phi M_{t+1} + (1-\phi) M_t}\frac{w_{t+1}}{w_t}\right]$$

$$(3) \qquad \frac{\phi M_{t+1} + (1-\phi) M_t}{\phi M_t + (1-\phi) M_{t-1}} = E_t\left[\left(\frac{w_{t+1}}{w_t}\right)^{1-\gamma}\right].$$

Despite the growth rate of money being clearly dependent on ϕ, the dynamics of optimal inflation rate are independent of $\phi \in (0, 1]$:

$$\frac{P_{t+1}}{P_t} = \frac{\phi M_{t+1} + (1-\phi) M_t}{\phi M_t + (1-\phi) M_{t-1}}\frac{w_t}{w_{t+1}} = E_t\left[\left(\frac{w_{t+1}}{w_t}\right)^{1-\gamma}\right]\frac{w_t}{w_{t+1}}.$$

Thus all our conclusions regarding the properties of the optimal expected inflation remain valid. Note, however that $\phi = 0$ would make the optimal policy infeasible, because by assumption, M_t is determined before w_t is known, thus making it impossible for equation (3) to hold. Intuitively, if all of the newly injected money was given to the young (who were to hold this money till the next period), then this new money would not affect the current price level, simply because it would not enter the money market in the current period.

3. Proof of Proposition 1

The equation for the optimal money growth rate

$$\frac{M_{t+1}}{M_t} = E_t\left[\left(\frac{w_{t+1}}{w_t}\right)^{1-\gamma}\right]$$

can be further transformed as follows

$$\begin{aligned}
\ln\frac{M_{t+1}}{M_t} &= \ln E_t\left[\left(\frac{w_{t+1}}{w_t}\right)^{1-\gamma}\right] = (\gamma-1)\ln w_t + \ln E_t\left[w_{t+1}^{1-\gamma}\right] \\
&= (\gamma-1)\,\omega_t + \ln E_t\left[(\exp\left[\omega_{t+1}\right])^{1-\gamma}\right] \\
&= (\gamma-1)\,\omega_t + \ln E_t \exp\left[(1-\gamma)\,\rho\omega_t + (1-\gamma)\,\varepsilon_{t+1}\right] \\
&= (\gamma-1)\,(1-\rho)\,\omega_t + \ln E_t \exp\left[(1-\gamma)\,\varepsilon_{t+1}\right].
\end{aligned}$$

The assumption of log-normality of the endowment process, implies that

$$\ln E_t \exp\left[(1-\gamma)\,\varepsilon_{t+1}\right] = \frac{(1-\gamma)^2\sigma^2}{2},$$

which implies equation (16)

$$m_{t+1} - m_t = \frac{(1-\gamma)^2\sigma^2}{2} + (1-\rho)(\gamma-1)\omega_t.$$

Similarly, for optimal inflation we obtain

$$\begin{aligned}
\ln\frac{P_{t+1}}{P_t} &= \ln\left(\frac{M_{t+1}}{M_t}\frac{w_t}{w_{t+1}}\right) = m_{t+1} - m + \omega_t - \omega_{t+1} \\
&= \frac{(1-\gamma)^2\sigma^2}{2} + (1-\rho)(\gamma-1)\omega_t + (1-\rho)\,\omega_t - \varepsilon_{t+1}
\end{aligned}$$

which implies equation (17)

$$p_{t+1} - p_t = \frac{(1-\gamma)^2\sigma^2}{2} + \gamma(1-\rho)\omega_t - \varepsilon_{t+1}.$$

The equation (17) in turn, implies equation (18) :

$$E_t\left[p_{t+1}-p_t\right]=\frac{(1-\gamma)^2\sigma^2}{2}+\gamma(1-\rho)\omega_t,$$

which immediately confirms all of the stated properties of optimal expected inflation.

4. Proof of Proposition 2

1. First we prove that the optimal monetary policy implies a positive average inflation. Since the logarithm is a concave function, by Jensen's inequality we have

$$\begin{aligned}
& \ln M_{t+1}-\ln M_t=\ln E_t\left[\left(\frac{A_{t+1}K_t^\alpha-K_{t+1}}{A_tK_{t-1}^\alpha-K_t}\right)^{-\gamma}\frac{x_{t+1}}{x_t}\right] \\
\geq\ & E_t\left[\left\{\ln x_{t+1}-\gamma\ln\left(A_{t+1}K_t^\alpha-K_{t+1}\right)\right\}-\left\{\ln x_t-\gamma\ln\left(A_tK_{t-1}^\alpha-K_t\right)\right\}\right].
\end{aligned}$$

Taking the unconditional expectation on both sides of the above inequality and noting that $\left(\ln x_t-\gamma\ln\left(A_tK_{t-1}^\alpha-K_t\right)\right)$ has a stationary distribution, we obtain

$$\begin{aligned}
& E\left[\ln M_{t+1}-\ln M_t\right] \\
\geq\ & E\left\{\ln x_{t+1}-\gamma\ln\left(A_{t+1}K_t^\alpha-K_{t+1}\right)\right\}-E\left\{\ln x_t-\gamma\ln\left(A_tK_{t-1}^\alpha-K_t\right)\right\}=0.
\end{aligned}$$

Furthermore, when the term $\left(\ln x_t-\gamma\ln\left(A_tK_{t-1}^\alpha-K_t\right)\right)$ is stochastic, the inequality above is strict, implying a positive average inflation. Whether this term is stochastic depends on the relative risk aversion parameter γ.

If the utility function is logarithmic in consumption, $\gamma=1$, then the optimal capital investment is proportional to output $K_{t+1}=\alpha A_{t+1}K_t^\alpha$ and, similar to the endowment economy, the optimal money stock is constant:

$$\begin{aligned}
\ln M_{t+1} - \ln M_t &= \ln E_t \left[\left(\frac{A_{t+1} K_t^{\alpha} - K_{t+1}}{A_t K_{t-1}^{\alpha} - K_t} \right)^{-1} \frac{x_{t+1}}{x_t} \right] \\
&= \ln E_t \left[\left(\frac{A_{t+1} K_t^{\alpha} - K_{t+1}}{A_t K_{t-1}^{\alpha} - K_t} \right)^{-1} \frac{(1+\tau)\frac{\beta}{1+\beta}\left(A_{t+1}K_t^{\alpha} - K_{t+1}\right) - \alpha A_{t+1} K_t^{\alpha}}{(1+\tau)\frac{\beta}{1+\beta}\left(A_t K_{t-1}^{\alpha} - K_t\right) - \alpha A_t K_{t-1}^{\alpha}} \right] \\
&= \ln E_t \left[\left(\frac{(1-\alpha) A_{t+1} K_t^{\alpha}}{(1-\alpha) A_t K_{t-1}^{\alpha}} \right)^{-1} \frac{\left[(1+\tau)\frac{\beta(1-\alpha)}{1+\beta} - \alpha\right] A_{t+1} K_t^{\alpha}}{\left[(1+\tau)\frac{\beta(1-\alpha)}{1+\beta} - \alpha\right] A_t K_{t-1}^{\alpha}} \right] = 0
\end{aligned}$$

For all other values of $\gamma > 0$ the inequality $E\left[\ln M_t - \ln M_{t-1}\right] \geq 0$ is strict, the optimal inflation rate is positive and (at least in all numerical simulations) is increasing with uncertainty.

2. Next, to verify the second property of the optimal monetary policy, we find a first-order approximated solution. Defining the aggregate consumption as $C_t = A_t K_{t-1}^{\alpha} - K_t$ we can state the equilibrium equations as

$$\begin{aligned}
C_t &= A_t K_{t-1}^{\alpha} - K_t \ , \\
C_t^{-\gamma} &= E_t \left\{ C_{t+1}^{-\gamma} \alpha A_{t+1} K_t^{\alpha - 1} \right\} .
\end{aligned}$$

The log-linear approximation of these equations around a non-stochastic steady state gives

$$\begin{aligned}
\bar{C} c_t &= \bar{K}^{\alpha} a_t + \alpha \bar{K}^{\alpha} k_{t-1} - \bar{K} k_t \\
-\gamma c_t &= E_t \left\{ -\gamma c_{t+1} + a_{t+1} - (1-\alpha) k_t \right\} ,
\end{aligned}$$

where $\bar{K} = \alpha^{\frac{1}{1-\alpha}}$, $\bar{C} = \bar{K}^{\alpha} - \bar{K}$, and $c_t = \ln\left(\frac{C_t}{\bar{C}}\right)$, $k_t = \ln\left(\frac{K_t}{\bar{K}}\right)$, $a_t = \ln A_t$. The solution of this system is

$$\begin{aligned}
c_t &= \frac{1}{1-\alpha}\left(a_t + \alpha k_{t-1} - \alpha k_t\right) \ , \\
k_{t+1} &= \phi_1 k_t + \phi_2 a_{t+1},
\end{aligned}$$

where

$$\phi_1 = \frac{\left[2\gamma\alpha + (1-\alpha)^2\right] - \sqrt{\left[2\gamma\alpha + (1-\alpha)^2\right]^2 - 4\gamma^2\alpha^2}}{2\gamma\alpha},$$

$$\phi_2 = \frac{\gamma + \rho(1-\alpha) - \gamma\rho}{2\gamma\alpha - \gamma\alpha\phi_1 + (1-\alpha)^2 - \gamma\alpha\rho}.$$

Further, by using a guess-and-verify approach, we can show that in the log-linear solution, the expected inflation satisfies

$$E_t\pi_{t+1} = \eta_\pi \pi_t + \eta_y y_t + \eta_a a_{t-1},$$

where

$$\eta_\pi = \frac{(1-\alpha)(\phi_1 - \alpha\phi_2) + \alpha\rho}{1 - \alpha + \alpha\psi},$$

$$\eta_y = (1-\alpha)\phi_2 + \eta_\pi\psi - \rho,$$

$$\eta_a = \eta_\pi\rho(1-\psi),$$

and the parameter ψ is given by

$$\psi = \frac{1}{\bar{x}}\left[(1+\tau)\frac{\beta^{\frac{1}{\gamma}}}{1+\beta^{\frac{1}{\gamma}}}\frac{\left(\bar{K}^\alpha - \bar{K}\right)}{(1-\alpha)}(1-\alpha\phi_2) - \alpha\bar{K}^\alpha\right]$$

with $\bar{x}$ being the steady-state value of the optimal real money balances in equation (28).

3. Substituting the value of $\gamma = 1$ into the above equations, it is straightforward to verify that in the log-utility case the elasticity coefficients become

$$\eta_\pi = \alpha\rho,$$

$$\eta_y = (1-\alpha)(1-\rho) \text{ and}$$

$$\eta_a = 0.$$

Moreover, in this special case the log-linear solution is exact.

Table 1: Parameter values for model simulations.

Parameter	Values
Discount factor, β	0.96^{30}
Relative risk aversion, γ	0.5, 1, 1.5, 4
Capital share in production function, α	0.33

Table 2: The accuracy of the first-order approximated solution for optimal monetary policy.

Ω_1	$\gamma = 0.5$	$\gamma = 1.0$	$\gamma = 1.5$	$\gamma = 4$
$\rho = 0.1$	0.999	1	1	0.999
$\rho = 0.5$	0.998	1	0.999	0.992
$\rho = 0.9$	0.999	1	0.999	0.956

Table 3: The accuracy of approximating the optimal expected inflation with a linear inflation targeting rule.

Ω_2	$\gamma = 0.5$	$\gamma = 1.0$	$\gamma = 1.5$	$\gamma = 4$
$\rho = 0.1$	0.999	1	1	0.999
$\rho = 0.5$	0.997	1	0.999	0.991
$\rho = 0.9$	0.957	1	0.993	0.938

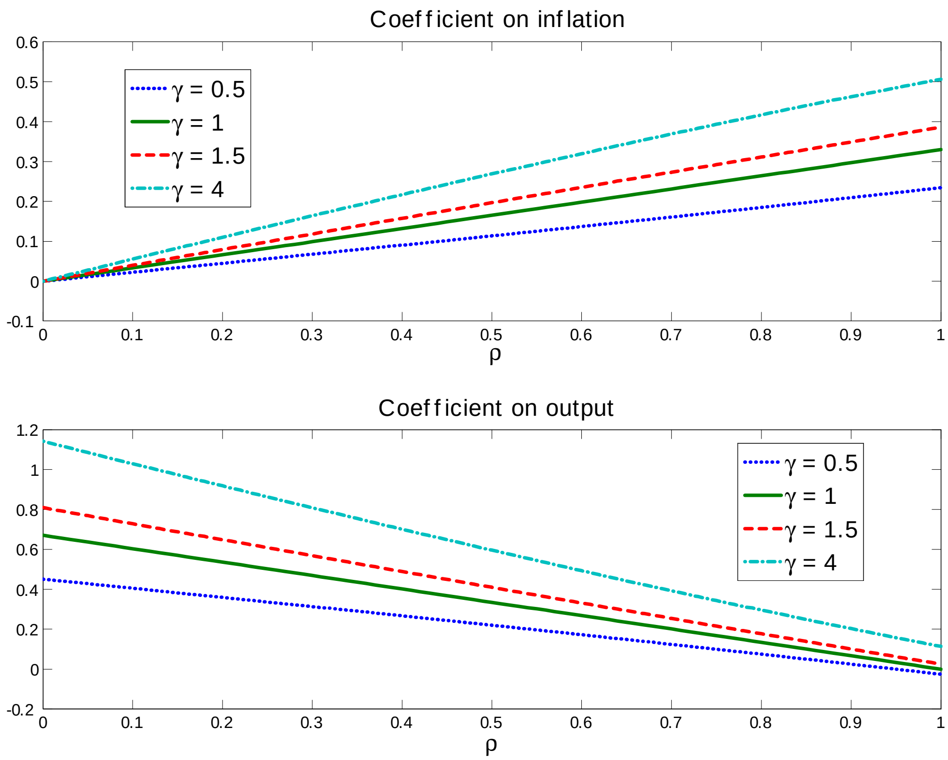

Figure 1: Top panel shows the elasticity of expected inflation to fluctuations in current inflation, for different values of persistense, ρ, and of risk aversion, γ. Lower panel shows the elasticity of expected inflation to fluctuations in current output, for different values of persistense, ρ, and of risk aversion, γ.

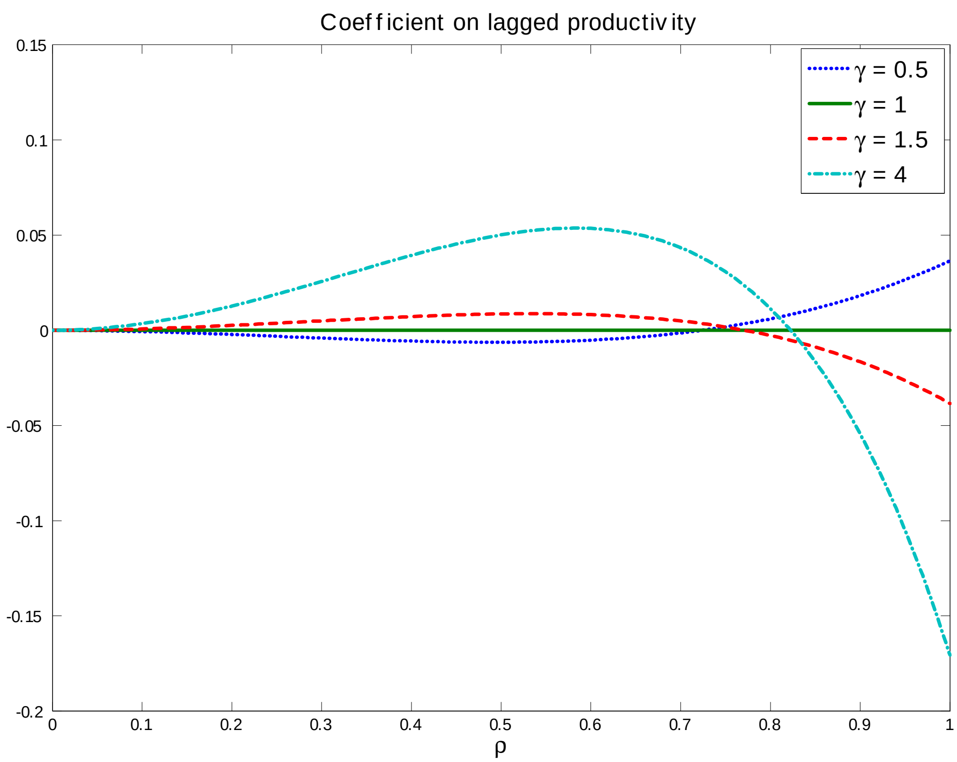

Figure 2: The figure shows the elasticity of expected inflation with respect to fluctuations in lagged productivity, for different values of persistense, ρ, and of risk aversion, γ.